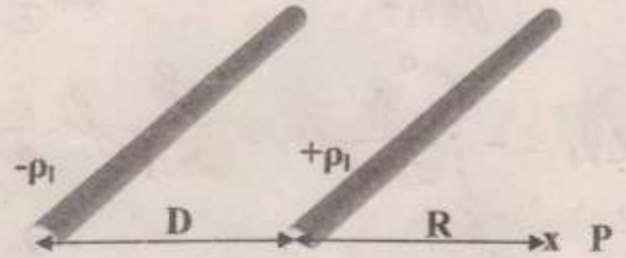
EXAMEN PARCIAL DE TEORÍA ELECTROMAGNETICA I DICIEMBRE/2010

NOMBRE: .... PARALELO: ....

1.- (30%) Se tienen dos conductores paralelos de sección transversal muy pequeña e infinitamente largos, con densidad lineal de carga $-\rho_l$ y $+\rho_l$ respectivamente, separados una distancia D. Calcular el potencial absoluto en el punto P ubicado a una distancia R de la línea de carga positiva.



$$\bar{E}_- = \frac{-\rho_l}{2\pi\varepsilon_0 x}\,\bar{a}_x$$

$$\bar{E}_+ = \frac{\rho_l}{2\pi\varepsilon_0 (x-D)}\,\bar{a}_x$$

$$\bar{E}_T = \frac{\rho_l}{2\pi\varepsilon_0}\left[\frac{1}{x-D} - \frac{1}{x}\right]\bar{a}_x$$

$$V_P = -\int_{a=\infty}^{b=D+R} \frac{\rho_l}{2\pi\varepsilon_0}\left[\frac{1}{x-D} - \frac{1}{x}\right]dx = -\frac{\rho_l}{2\pi\varepsilon_0}\Big[\ln(x-D) - \ln x\Big]_{a=\infty}^{b=D+R}$$

$$V_P = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{x}{x-D}\Bigg|_{\infty}^{D+R} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{1}{1-\frac{D}{x}}\Bigg|_{\infty}^{D+R}$$

$$V_P = \frac{\rho_l}{2\pi\varepsilon_0}\left[\ln\frac{D+R}{R} - \cancel{\ln 1}^{\,0}\right] \Rightarrow \boxed{V_P = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{D+R}{R}}$$

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Por suma de potenciales

$$V_{P(-)} = -\int_{a=\infty}^{b=D+R} -\frac{\rho_l}{2\pi\varepsilon_0 x}\,dx = \frac{\rho_l}{2\pi\varepsilon_0}\ln x\Big|_a^b = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{b}{a}$$

$$V_{P(+)} = -\int_{a=\infty}^{b=D+R} \frac{\rho_l}{2\pi\varepsilon_0 (x-D)}\,dx = -\frac{\rho_l}{2\pi\varepsilon_0}\ln(x-D)\Big|_a^b = \frac{\rho_l}{2\pi\varepsilon_0}\ln(x-D)\Big|_b^a$$

$$V_{P(+)} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a-D}{b-D} \qquad V_P = V_{P(+)} + V_{P(-)} = \frac{\rho_l}{2\pi\varepsilon_0}\left[\ln\frac{a-D}{b-D} + \ln\frac{b}{a}\right]$$

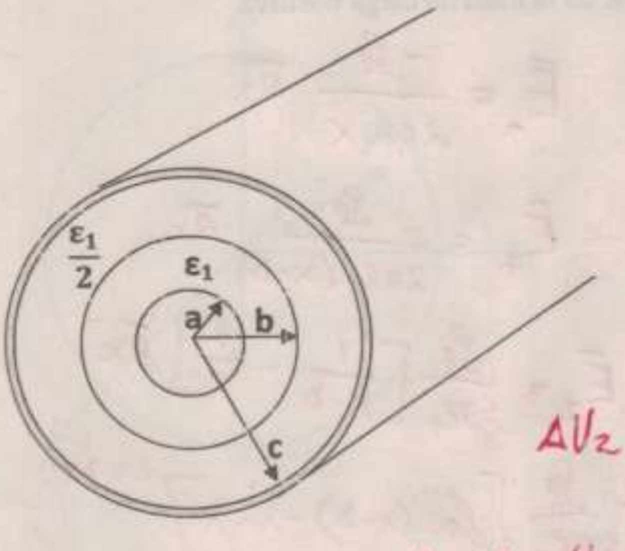
$$V_P = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{b}{a}\left(\frac{a-D}{b-D}\right) = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{b\left(1-\frac{D}{a}\right)}{b-D} \qquad \begin{matrix} a=\infty \\ b=D+R \end{matrix}$$

$$\boxed{V_P = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{D+R}{R}}$$

2.- Un cable coaxial de radio interior **a** y radio exterior **c**, tiene en su interior dos dieléctricos de permitividades ε_1 y $\frac{\varepsilon_1}{2}$, como indica la figura.

a) (20%) Calcular el valor del radio **b** de separación de los dos dieléctricos, para que la diferencia de potencial en cada dieléctrico sea igual.

b) (15%) Calcular la capacitancia por unidad de longitud del cable. (la respuesta no debe quedar expresada en términos del radio **b**)



$$D_1 = D_2 = D = \frac{\rho_l}{2\pi r}\bar{a}_r$$

$$E_1 = \frac{\rho_l}{2\pi\varepsilon_1 r}\bar{a}_r \qquad E_2 = \frac{\rho_l}{2\pi\frac{\varepsilon_1}{2}r}\bar{a}_r$$

$$\Delta V_1 = -\int_b^a \frac{\rho_l}{2\pi\varepsilon_1 r}dr = \frac{\rho_l}{2\pi\varepsilon_1}\ln\frac{b}{a}$$

$$\Delta V_2 = -\int_c^b \frac{\rho_l}{\pi\varepsilon_1 r}dr = \frac{\rho_l}{\pi\varepsilon_1}\ln\frac{c}{b}$$

$\Delta V_1 = \Delta V_2$ (condición del problema)

$$\frac{\cancel{\rho_l}}{2\cancel{\pi\varepsilon_1}}\ln\frac{b}{a} = \frac{\cancel{\rho_l}}{\cancel{\pi\varepsilon_1}}\ln\frac{c}{b} \Rightarrow \ln\frac{b}{a} = 2\ln\frac{c}{b} = \ln\left(\frac{c}{b}\right)^2$$

$$\frac{b}{a} = \frac{c^2}{b^2} \Rightarrow b^3 = ac^2 \qquad \boxed{b = \sqrt[3]{ac^2}}$$

b) $Q = C_1\Delta V_1 \quad Q = C_2\Delta V_2 \quad$ Si $\Delta V_1 = \Delta V_2 \quad C_1 = C_2 \quad C_T = \frac{C_1}{2} = \frac{C_2}{2}$

$$\Delta V_1 = \frac{\rho_l}{2\pi\varepsilon_1}\ln\frac{b}{a} \Rightarrow C_1/l = \frac{2\pi\varepsilon_1}{\ln\frac{b}{a}} = \frac{2\pi\varepsilon_1}{\ln\frac{(ac^2)^{1/3}}{a}} = \frac{2\pi\varepsilon_1}{\ln\frac{a^{1/3}c^{2/3}}{a}}$$

$$C_1/l = \frac{2\pi\varepsilon_1}{\ln\frac{c^{2/3}}{a^{2/3}}} = \frac{2\pi\varepsilon_1}{\ln\left(\frac{c}{a}\right)^{2/3}} = \frac{2\pi\varepsilon_1}{\frac{2}{3}\ln\frac{c}{a}} = \frac{3\pi\varepsilon_1}{\ln\frac{c}{a}}$$

$$\boxed{C_T/l = \frac{3\pi\varepsilon_1}{2\ln\frac{c}{a}}}$$

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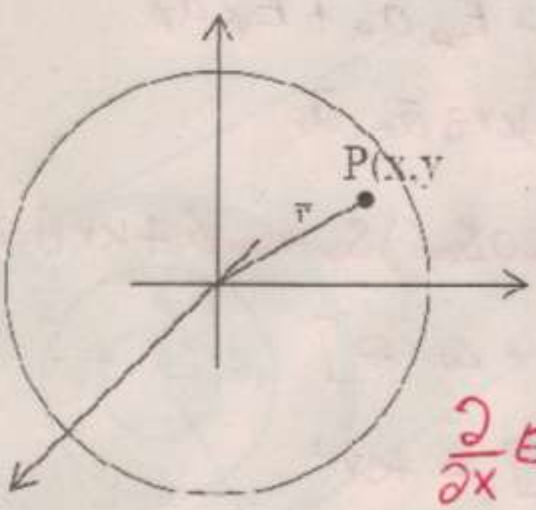
$$\Delta V_2 = \frac{\rho_l}{\pi\varepsilon_1}\ln\frac{c}{b} \Rightarrow C_2/l = \frac{\pi\varepsilon_1}{\ln\frac{c}{b}} = \frac{\pi\varepsilon_1}{\ln\frac{c}{(ac^2)^{1/3}}}$$

$$C_2/l = \frac{\pi\varepsilon_1}{\ln\frac{c^{1/3}}{a^{1/3}}} = \frac{\pi\varepsilon_1}{\ln\left(\frac{c}{a}\right)^{1/3}}$$

$$C_2/l = \frac{\pi\varepsilon_1}{\frac{1}{3}\ln\frac{c}{a}} = \frac{3\pi\varepsilon_1}{\ln\frac{c}{a}} = C_1/l$$

3.- (35%) En un punto **P(x,y,z)** de una región del espacio, hay campo eléctrico
$\mathbf{E} = krx\ \mathbf{a}_x + kry\ \mathbf{a}_y + krz\ \mathbf{a}_z$ donde **k** es una constante y **r** es la distancia del punto **P** respecto del origen de coordenadas.

Calcular la carga total contenida en el volumen limitado por una superficie esférica de radio *R* centrada en el origen.



$$r = \sqrt{x^2+y^2+z^2}$$

$$\nabla \cdot \bar{E} = \frac{\rho_v}{\varepsilon_0} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\frac{\partial}{\partial x} E_x = \frac{\partial}{\partial x} k(x^2+y^2+z^2)^{1/2} x$$

$$\frac{\partial}{\partial x} E_x = k\left[\underbrace{(x^2+y^2+z^2)^{1/2}}_{r} + x\,\frac{1}{2}(x^2+y^2+z^2)^{-1/2} 2x\right]$$

$$\frac{\partial}{\partial x} E_x = k\left[r + \frac{x^2}{r}\right]$$

En forma semejante:

$$\frac{\partial}{\partial y} E_y = k\left[r + \frac{y^2}{r}\right] \qquad \frac{\partial}{\partial z} E_z = k\left[r + \frac{z^2}{r}\right]$$

$$\frac{\rho_v}{\varepsilon_0} = k\left[r + \frac{x^2}{r}\right] + k\left[r + \frac{y^2}{r}\right] + k\left[r + \frac{z^2}{r}\right] = k\left[3r + \frac{\overbrace{x^2+y^2+z^2}^{r^2}}{r}\right]$$

$$\frac{\rho_v}{\varepsilon_0} = 4kr \qquad \boxed{\rho_v = 4k\varepsilon_0 r}$$

$$Q = \int_0^R \rho_v\, d\mathrm{vol} = \int_0^R (4k\varepsilon_0 r)\, 4\pi r^2 dr = 16k\pi\varepsilon_0 \int_0^R r^3 dr$$

$$Q = 16k\pi\varepsilon_0 \left.\frac{r^4}{4}\right|_0^R = \boxed{4\pi\varepsilon_0 k R^4}$$

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3) Otra manera de resolverlo:

$$\oint \bar{E}\cdot d\bar{s} = \frac{Q}{\varepsilon_0} \qquad d\bar{s} = r^2 \operatorname{Sen}\theta \, d\theta \, d\phi \, \bar{a}_r$$

$$x = r \operatorname{Sen}\theta \operatorname{Cos}\phi \quad y = r \operatorname{Sen}\theta \operatorname{Sen}\phi$$

$$z = r \operatorname{Cos}\theta$$

$$\bar{E} = krx\,\bar{a}_x + kry\,\bar{a}_y + krz\,\bar{a}_z = E_r\,\bar{a}_r + E_\theta\,\bar{a}_\theta + E_\phi\,\bar{a}_\phi$$

$$E_r = \bar{E}\cdot\bar{a}_r = krx\,\bar{a}_x\cdot\bar{a}_r + kry\,\bar{a}_y\cdot\bar{a}_r + krz\,\bar{a}_z\cdot\bar{a}_r$$

$$= kr(r\operatorname{Sen}\theta\operatorname{Cos}\phi)(\operatorname{Sen}\theta\operatorname{Cos}\phi) + kr(r\operatorname{Sen}\theta\operatorname{Sen}\phi)\operatorname{Sen}\theta\operatorname{Sen}\phi + kr(r\operatorname{Cos}\theta)\operatorname{Cos}\theta$$

$$= kr^2\left[\operatorname{Sen}^2\theta\operatorname{Cos}^2\phi + \operatorname{Sen}^2\theta\operatorname{Sen}^2\phi + \operatorname{Cos}^2\theta\right]$$

$$E_r = kr^2\left[\operatorname{Sen}^2\theta(\operatorname{Cos}^2\phi + \operatorname{Sen}^2\phi) + \operatorname{Cos}^2\theta\right] = kr^2$$

$$E_\theta = \bar{E}\cdot\bar{a}_\theta = krx\,\bar{a}_x\cdot\bar{a}_\theta + kry\,\bar{a}_y\cdot\bar{a}_\theta + krz\,\bar{a}_z\cdot\bar{a}_\theta$$

$$= kr(r\operatorname{Sen}\theta\operatorname{Cos}\phi)\operatorname{Cos}\theta\operatorname{Cos}\phi + kr(r\operatorname{Sen}\theta\operatorname{Sen}\phi)\operatorname{Cos}\theta\operatorname{Sen}\phi + kr(r\operatorname{Cos}\theta)(-\operatorname{Sen}\theta)$$

$$= kr^2\operatorname{Sen}\theta\operatorname{Cos}\theta\operatorname{Cos}^2\phi + kr^2\operatorname{Sen}\theta\operatorname{Cos}\theta\operatorname{Sen}^2\phi - kr^2\operatorname{Sen}\theta\operatorname{Cos}\theta$$

$$= kr^2\left[\operatorname{Sen}\theta\operatorname{Cos}\theta(\operatorname{Cos}^2\phi + \operatorname{Sen}^2\phi) - \operatorname{Sen}\theta\operatorname{Cos}\theta\right] = 0$$

$$E_\phi = \bar{E}\cdot\bar{a}_\phi = krx\,\bar{a}_x\cdot\bar{a}_\phi + kry\,\bar{a}_y\cdot\bar{a}_\phi + krz\,\bar{a}_z\cdot\bar{a}_\phi$$

$$= kr(r\operatorname{Sen}\theta\operatorname{Cos}\phi)(-\operatorname{Sen}\phi) + kr(r\operatorname{Sen}\theta\operatorname{Sen}\phi)\operatorname{Cos}\phi + kr(r\operatorname{Cos}\theta)(0)$$

$$= kr^2\left[-\operatorname{Sen}\theta\operatorname{Sen}\phi\operatorname{Cos}\phi + \operatorname{Sen}\theta\operatorname{Sen}\phi\operatorname{Cos}\phi\right] = 0$$

$$\boxed{\bar{E} = kr^2\,\bar{a}_r} \qquad \oint kr^2\,\bar{a}_r\cdot r^2\operatorname{Sen}\theta\,d\theta\,d\phi\,\bar{a}_r = \frac{Q}{\varepsilon_0}$$

$$kR^4\int_0^{\pi}\operatorname{Sen}\theta\,d\theta\int_0^{2\pi}d\phi = kR^4\left[-\operatorname{Cos}\theta\right]_0^{\pi}\left[\phi\right]_0^{2\pi} = kR^4[1+1][2\pi]$$

$$4\pi kR^4 = \frac{Q}{\varepsilon_0} \Rightarrow \boxed{Q = 4\pi\varepsilon_0 kR^4}$$