APPLICATION OF GALERKIN'S METHOD
TO ONE-DIMENSIONAL TWO-PHASE FLUID FLOW

by

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INTRODUCTION

The problem of calculating the pressure and saturation distribution in a reservoir is of considerable importance in a simulation of a reservoir performance.

Water is injected at a well into a zone saturated primarily with oil. A watered-out zone develops around the injection well where high water saturation prevails. Farther away from the well, a transition zone is created where both fluids have a significant mobility, this "front", moves toward the producing well in response to the injection rate. Ahead of the front, the fluid moving is mainly oil being pushed by the water.

The Buckley-Leverett technique is known to solve this immiscible fluid displacement problem. Finite difference is also a common approach to solve the differential equations that describe the process. This thesis presents and examines results obtained by the application of Galerkin's method to one-dimensional, two-phase fluid flow in porous media. The approximation obtained by using Galerkin's method has been proven to be theoretically and numerically superior to the usual approximation by finite differences.[3] The application of the technique studied in this thesis may also be used for multiphase, multidimensional flow.

Convergence of the discrete numerical solution is shown for pressure and saturation as well as the gradients. Cubic piecewise-polynomial approximations are used in the model and oil and water relative permeabilities are written as quadratic functions of water saturation.

THE MATHEMATICAL MODEL

The flow of fluids through a porous media obeys Darcy's law, which states that the velocity in barrels per day per square foot of a homogeneous fluid is proportional to the fluid viscosity[1,2]

$$\bar{v} = -1.127 \frac{kk_r}{\mu} (\nabla P + \rho \nabla Z) \tag{1.1}$$

In this equation*, kk_r, the effective permeability, is the constant of proportionality expressed in Darcy units; the driving force, $(\nabla P + \rho\nabla Z)$ in psi/ft, and μ the fluid viscosity in centipoises.

In the quantity $(\nabla P + \rho\nabla Z)$, the term $\rho\nabla Z$, is the contribution of gravity forces which is neglected in this study, such that, the driving force is ∇P. With the above assumption, for the case of linear flow, equation (1.1) can be written as follows

$$v = -1.127 \frac{kk_r}{\mu} \frac{\partial P}{\partial x} \tag{1.2}$$

If oil and water are moving in the system, the last equation can be written for each phase to get:

$$v_o = -1.127 \frac{kk_{ro}}{\mu_o} \frac{\partial P}{\partial x} \tag{1.3}$$

$$v_w = -1.127 \frac{kk_{rw}}{\mu_w} \frac{\partial P}{\partial x} \tag{1.4}$$

* Definitions and symbols are given in APPENDIX A.

where the subscript "o" refers to oil and "w" to water. In this study the pressure in the oil and water zone is considered the same, that is, the capillary forces are assumed negligible and are not included.

The continuity condition for each phase is given by

$$-\frac{\partial v_o}{\partial x} = \frac{\phi}{5.615}\frac{\partial S_o}{\partial t} \quad (1.5)$$

$$-\frac{\partial v_w}{\partial x} = \frac{\phi}{5.615}\frac{\partial S_w}{\partial t} \quad (1.6)$$

where the constant 5.615 appears as a result of the field units used. After substitution of equation (1.3) into (1.5), the basic equation that describes the linear flow of oil in a porous media is obtained:

$$\frac{\partial}{\partial x}\left(\frac{kk_{ro}}{\mu_o}\frac{\partial P}{\partial x}\right) = \frac{\phi}{6.328}\frac{\partial S_o}{\partial t} \quad (1.7)$$

and similarly for the water phase

$$\frac{\partial}{\partial x}\left(\frac{kk_{rw}}{\mu_w}\frac{\partial P}{\partial x}\right) = \frac{\phi}{6.328}\frac{\partial S_w}{\partial t} \quad (1.8)$$

If only oil and water are saturating the pores of the rock, we can say

$$S_o + S_w = 1 \quad (1.9)$$

such that

$$\frac{\partial S_o}{\partial t} = -\frac{\partial S_w}{\partial t} = \frac{\partial S}{\partial t} \quad (1.10)$$

Therefore the equations that describe the mathematical model are:

$$\frac{\partial}{\partial x}\left(\frac{kk_{ro}}{\mu_o}\frac{\partial P}{\partial x}\right) = -\frac{\phi}{6.328}\frac{\partial S}{\partial t} \qquad (1.11)$$

$$\frac{\partial}{\partial x}\left(\frac{kk_{rw}}{\mu_w}\frac{\partial P}{\partial x}\right) = \frac{\phi}{6.328}\frac{\partial S}{\partial t} \qquad (1.12)$$

Initial and Boundary Conditions

There are several conditions that can be imposed to the model and after some study the following were selected.

Boundary Conditions:

1. The water injection rate is constant. From Darcy's law, this condition leads to a constant pressure gradient at the inlet edge of the system.
2. The pressure at the producing face of the model is held constant at the initial value P_i.
3. The injection face is considered to be flooded instantaneously; that is, the oil saturation on that face is the residual oil saturation.
4. The saturation gradient, $\frac{\partial S}{\partial x}$, at x = 0 is kept at zero.

Initial Conditions:

1. The pressure along the model is initially P_i, a constant.
2. The existent water is initially at its irreducible value S_{wi}.

Mathematically, the initial and boundary conditions can be written as follows:

$$Q_{in} = \text{constant} \quad \text{or}$$

$$\frac{\partial}{\partial x}\left[P(o,t)\right] = -\left.\frac{q_{in}\mu_w}{1.127\ kA\ k_{rw}}\right|_{x=o}$$

$$P(L,t) = P_i$$

$$S(o,t) = 1 - S_{or}$$

$$\frac{\partial}{\partial x}\left[S(o,t)\right] = 0$$

$$P(x,o) = P_i$$

$$S(x,o) = S_{wi}$$

GALERKIN METHOD OF SOLUTION

For the sake of completeness, let us write the mathematical statement of the problem in the following way:

Solve

$$L_1(P,S) = \frac{\partial}{\partial x}\left(\frac{kk_{ro}}{\mu_o}\frac{\partial P}{\partial x}\right) + \frac{\phi}{6.328}\frac{\partial S}{\partial t} = 0 \qquad (2.1)$$

$$L_2(P,S) = \frac{\partial}{\partial x}\left(\frac{kk_{rw}}{\mu_w}\frac{\partial P}{\partial S}\right) \frac{\phi}{6.328}\frac{\partial S}{\partial t} = 0 \qquad (2.2)$$

subject to the boundary and initial conditions stated in (1.13). The solution of these nonlinear partial parabolic differential equations is to be found by Galerkin method.[3,4]

Let T denote the class of all real valued piecewise continuously differentiable functions in space on the region $R\{(x,t)/0<x<L\}$. Let T_p be a p-dimensional subspace of T spanned by the p basis functions $w_k(x)$, $k = 1$ to p. In the region R we seek a solution to equations (2.1) and (2.2) of the form

$$P^* = \sum_{k=1}^{p} A_k(t)\ w_k(x) \qquad (2.3)$$

$$S^* = \sum_{k=1}^{p} B_k(t)\ s_k(x) \qquad (2.4)$$

where the coefficients A_k and B_k are determined by the conditions that $L_1(P^*,S^*)$ and $L_2(P^*,S^*)$ each be orthogonal to T_p for all values of $t > 0$, and that P^* and S^* satisfy the boundary conditions,

$$\int_o^L L_1(P^*,S^*)\; w_j(x)\; dx = 0 \qquad (2.5)$$

$$\int_o^L L_2(P^*,S^*)\; w_j(x)\; dx = 0 \qquad (2.6)$$

Using equations (2.1) and (2.2) we have

$$\int_o^L \left[\frac{\partial}{\partial x}\left(\frac{kk_{ro}}{\mu_o}\frac{\partial P^*}{\partial x}\right) + \frac{\phi}{6.328}\frac{\partial S^*}{\partial t}\right] w_j(x)\; dx = 0 \qquad (2.7)$$

$$\int_o^L \left[\frac{\partial}{\partial x}\left(\frac{kk_{rw}}{\mu_w}\frac{\partial P^*}{\partial x}\right) - \frac{\phi}{6.328}\frac{\partial S^*}{\partial t}\right] w_j(x)\; dx = 0 \qquad (2.8)$$

for $j = 1, 2, 3, \ldots, p$

After straightforward integration by parts

$$\left[\frac{kk_{ro}}{\mu_o}\frac{\partial P^*}{\partial x}\, w_j(x)\right]_o^L - \int_o^L \frac{kk_{ro}}{\mu_o}\frac{\partial P^*}{\partial x}\, w_j'(x)\; dx$$

$$+ \frac{\phi}{6.328}\int_o^L \frac{\partial S^*}{\partial t}\, w_j(x)\; dx = 0 \qquad (2.9)$$

$$\left[\frac{kk_{rw}}{\mu_w}\frac{\partial P^*}{\partial x}\, w_j(x)\right]_o^L - \int_o^L \frac{kk_{rw}}{\mu_w}\frac{\partial P^*}{\partial x}\, w_j'(x)\; dx$$

$$- \frac{\phi}{6.328}\int_o^L \frac{\partial S^*}{\partial t}\, w_j(x)\; dx = 0 \qquad (2.10)$$

for $j = 1, 2, 3, \ldots, p$

In these last two equations we have:

$$\frac{\partial P^*}{\partial x} = \sum_{k=1}^{p} A_k w_k'$$

$$\frac{\partial S^*}{\partial x} = \sum_{k=1}^{p} B_k w_k'$$

$$\frac{\partial P^*}{\partial t} = \sum_{k=1}^{p} A_k' w_k$$

$$\frac{\partial S^*}{\partial t} = \sum_{k=1}^{p} B_k' w_k \qquad (2.11)$$

Substitution of equations (2.11) into (2.9) and (2.10) gives:

$$\left[\frac{kk_{ro}}{\mu_o}\left(\sum_{k=1}^{p} A_k w_k'\right) w_j\right]_o^L - \int_o^L \left[\frac{kk_{ro}}{\mu_o}\left(\sum_{k=1}^{p} A_k w_k'\right) w_j'\right] dx$$

$$+ \frac{\phi}{6.328}\int_o^L \left[\left(\sum_{k=1}^{p} B_k' w_k\right) w_j\right] dx = 0 \qquad (2.12)$$

and

$$\left[\frac{kk_{rw}}{\mu_w}\left(\sum_{k=1}^{p} A_k w_k'\right) w_j\right]_o^L - \int_o^L \left[\frac{kk_{rw}}{\mu_w}\left(\sum_{k=1}^{p} A_k w_k'\right) w_j'\right] dx$$

$$- \frac{\phi}{6.328}\int_o^L \left[\left(\sum_{k=1}^{p} B_k' w_k\right) w_j\right] dx = 0 \qquad (2.13)$$

for $j = 1, 2, 3, \ldots, p$

Smooth bi-cubic basis functions[5] have been used to solve the problem and their definitions and graphs are given in appendix B. Note that the first term of equations (2.12) and (2.13) has a value other than zero only for j = 1 and j = p; otherwise they vanish due to the definition of the basis functions.

Let n be the number of intervals in which the length L of the model is divided, such that $\Delta x = \frac{L}{n}$, then equations (2.12) and (2.13), each one constitutes a set of 2n + 2 equations to be solved simultaneously. These equations can be written in matrix notation as follows

$$C\overline{A} + D \frac{d\overline{B}}{dt} = \overline{G} \qquad (2.14)$$

$$E\overline{A} + F \frac{d\overline{B}}{dt} = \overline{H} \qquad (2.15)$$

<u>Structure of the Matrices</u>

The elements of every one of the matrices are known from the evaluation of the integrals which will be later discussed. Before taking into consideration any of the boundary conditions, the structure of matrices C, D, E and F is as follows

$$\begin{vmatrix} [J1] & [R1] & & & & 0 \\ [L2] & [J2] & [R2] & & & \\ & [L3] & [J3] & [R3] & & \\ & & \ddots & \ddots & \ddots & \\ & & [Ln] & [Jn] & [Rn] & \\ 0 & & & & [L(n+1)] & [J(n+1)] \end{vmatrix}$$

This is a square block tridiagonal matrix and every row is the result of writing equations (2.12) and (2.13) at a grid point. Each of the L's, J's and R's is a two by two matrix. The L matrices are made up by the coupling of the coefficients at the i^{th} grid point with those at the $(i-1)^{th}$ point. The J matrices result from the coupling of the coefficients at the i^{th} point with themselves, and similarly, the R matrices are made up by the coupling of the coefficients at the i^{th} grid point with the ones at the $(i+1)^{th}$ point.

Let us establish that 2i-1 is the subscript on the unknown pressure at the i^{th} grid point and 2i is the subscript on the pressure gradient at the same point, such that i takes values from one to n+1.

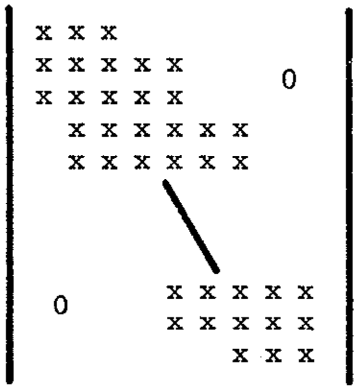
Matrices C and E

These are the two matrices associated with $\overline{A}$ and there are two boundary conditions related to this vector:

$$\frac{\partial}{\partial x}\,[P(o,t)] = \text{constant}$$

$$P(L,t) = P_i$$

Therefore, A_2 and A_{2n+1} are known and the corresponding equations in the system (2.14) and (2.15) are pulled out, so that we are left with the following structure for matrices C and E



where 'x' means a non-zero element.

Matrices D and F

These matrices are associated with $\frac{d\overline{B}}{dt}$ and we also have two boundary conditions related to saturation, namely:

$$\frac{\partial}{\partial x}\,[S(o,t)] = 0$$

$$S(o,t) = 1 - S_{or}$$

This means that B_1 and B_2 are known and after taking out the corresponding equations to these coefficients, the following structure results

$$\left|\begin{array}{ccccccccccc}
x & x & x & x & & & & & & & \\
x & x & x & x & & & & & 0 & & \\
x & x & x & x & x & x & & & & & \\
x & x & x & x & x & x & & & & & \\
 & & x & x & x & x & x & x & & & \\
 & & x & x & x & x & x & x & & & \\
 & & & & & \ddots & & & & & \\
 & & & & & x & x & x & x & x & x \\
 & & 0 & & & x & x & x & x & x & x \\
 & & & & & & & x & x & x & x \\
 & & & & & & & x & x & x & x
\end{array}\right|$$

<u>Vectors G and H</u>

As a result of the boundary conditions applied, these vectors are made up of some constant elements resulting from the product of known elements of $\overline{A}$ and $\overline{B}$ with the corresponding matrix elements. The structure of both vectors G and H is

$$\left|\begin{array}{c}
x \\ x \\ x \\ o \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ o \\ x \\ x
\end{array}\right|$$

EVALUATION OF INTEGRALS AND SOLUTION OF THE SYSTEM OF EQUATIONS

Evaluation of Integrals

For the sake of simplicity let us make

$$M = \frac{kk_{ro}}{\mu_o}$$

and

$$N = \frac{kk_{rw}}{\mu_w} \qquad (3.1)$$

Matrix C comes from the evaluation of

$$- \int_o^L M \left(\sum_{k=1}^{p} A_k w'_k \right) w'_j \, dx \qquad (3.2)$$

in equation (2.12) and similarly, matrix E comes from evaluating

$$- \int_o^L N \left(\sum_{k=1}^{p} A_k w'_k \right) w'_j \, dx \qquad (3.3)$$

in equation (2.13). The next step is to show how the integral of equation (3.2) is evaluated. The evaluation of (3.3) is basically the same with k_{rw} instead of k_{ro}. At the i^{th} grid point we have the situation shown in figure 1. Note that:

A_{2i-3} = pressure at i-1

A_{2i-2} = pressure gradient at i-1

A_{2i-1} = pressure at i

A_{2i} = pressure gradient at i

A_{2i+1} = pressure at i+1

A_{2i+2} = pressure gradient at i+1

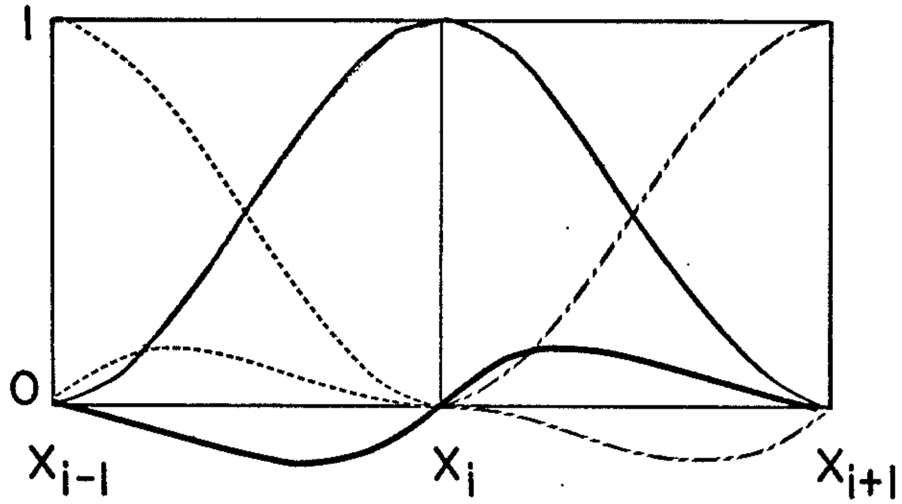
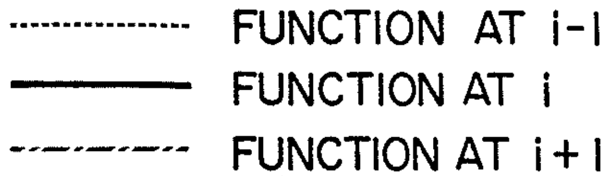


FIGURE 1



$$- \int_{o}^{L} M \left(\sum_{k=1}^{p} A_k W'_k \right) w_j dx$$

$$= - \int_{x_{i-1}}^{x_i} M \left(\sum_{k=1}^{p} A_k w'_k \right) w'_j \, dx - \int_{x_i}^{x_{i+1}} M \left(\sum_{k=1}^{p} A_k w'_k \right) w'_j \, dx \qquad (3.4)$$

$$= - \int_{x_{i-1}}^{x_i} M \left[A_{2i-3} f'_1 + A_{2i-2} f'_2 + A_{2i-1} f'_3 + A_{2i} f'_4 \right] f'_{m+2} dx$$

$$- \int_{x_i}^{X_{i+1}} M \left[A_{2i-1} f'_1 + A_{2i} f'_2 + A_{2i+1} f'_3 + A_{2i+2} f'_4 \right] f'_m dx$$

for m=1,2

Now if we order the terms involving the pressure with those involving the pressure gradients we get the L, J and R two by two matrices referred to in (2.16) as shown in appendix C. Each has elements of the form:

$$\int M \; Q(x) \; dx \qquad (3.6)$$

where Q(x) is a fourth degree polynomial in x resulting from the product of two derivatives of the basis functions. The relative permeabilities are expressed as second degree polynomials in saturation

$$k_{ro} = z_1 + z_2 S + z_3 S^2$$

$$k_{rw} = z_4 + z_5 S + z_6 S^2$$

The coefficients used in this study are:

$z_1 = 1.590355$

$z_2 = -3.742012$

$z_3 = 2.201183$

$z_4 = 5.680473 * 10^{-2}$

$z_5 = -0.560473$

$z_6 = 1.4201183$

These coefficients here are determined by fitting a quadratic through the two end points of the data of Table 1 and requiring a slope near zero on the abscissa. The saturation is a third degree polynomial in x, therefore the relative permeabilities are of sixth degree in the same variable; this means that the integrand in equation (3.6) is a tenth degree polynomial in x. In this study the relative permeabilities are evaluated at the beginning of every time step and considered constant during the time step.

Solution of the System of Equations

A great deal of work was spent in solving the system of equations given by (2.14) and (2.15). Many schemes were tried in searching for a solution of those equations, and finally a Gauss-Jordan type of solution was used.[6] A Fortran subroutine was written so that only the nonzero elements had to be stored in a p x 6 matrix. The reduction of a matrix, say for instance c, to an upper triangular form was done by an algorithm of the following form.

$$C_{i,j} = C_{i,j} \times \frac{C_{i-1,1}}{C_{i,1}} - C_{i-1,j} \qquad j = 1,2,\ldots 6$$

$$C_{i-1,j} = C_{i-1,j} \times \frac{C_{i-2,3}}{C_{i-1,1}} - C_{i-2,j+2} \qquad j = 1,2,3,4$$

$$C_{i-1,j} = C_{i-1,j} \times \frac{C_{i-2,3}}{C_{i-1,1}} \qquad j = 5,6$$

$$C_{i,j} = C_{i,j} \times \frac{C_{i-1,2}}{C_{i,2}} - C_{i-1,j} \qquad j = 2,3,4,5,6$$

$$C_{i-1,j} = C_{i-1,j} \times \frac{C_{i-2,4}}{C_{i-1,2}} - C_{i-2,j+2} \qquad j = 2,3,4$$

$$C_{i-1,j} = C_{i-1,j} \times \frac{C_{i-2,4}}{C_{i-1,2}} \qquad j = 5,6$$

$$C_{i,j} = C_{i,j} \times \frac{C_{i-1,3}}{C_{i,3}} - C_{i-1,j} \qquad j = 3,4,5,6$$

All the equations above are for $i = 2, 4, 6 \ldots p$.

A similar algorithim was used to manipulate the right hand side of each equation.

RESULTS AND DISCUSSION

Before solving the equations that describe the mathematical model for this study, the capillary forces were taken into consideration giving a more complex problem to solve, in this case, the pressures in the oil and water phase are different, being related to each other by the capillary pressure. Chapeau basis functions were used for that case with poor results. The linearity of the Chapeau functions made it difficult to fit the boundary conditions and they had to be approximated by finite differences.

The relative permeabilities have been approximated by a quadratic function of water saturation so that a quadrature scheme is not necessary to evaluate the integrals. A six point Gaussian quadrature[7] was used to check the results, and the polynomial approximation procedure was found to work well.

The Time Derivative

Successive overrelaxation[8,9] (SOR) was initially used to solve the equation (2.14) and (2.15). The time derivative, $\frac{d\bar{B}}{dt}$, was approximated by a backward difference such that the equations became

$$C\bar{A} + D\,\frac{(\bar{B}^{n+1} - \bar{B}^{n})}{\Delta t} = \bar{G} \qquad (4.18)$$

$$E\bar{A} - D\,\frac{(\bar{B}^{n+1} - \bar{B}^{n})}{\Delta t} = \bar{H} \qquad (4.19)$$

where $\bar{B}^{n}$ is a known vector. F has been replaced by its equivalent $-D$.

By adding (4.18) and (4.19) we get one equation in one unknown, the vector $\bar{A}$, and then SOR was used to solve it. $\bar{B}^{n+1}$ was calculated from equation (4.19) by the same iterative scheme. This scheme required a very small convergence criteria. As a result, many iterations were necessary each time step. This led to excessive computing time and accumulated round-off error. The best overrelaxation parameter(9) was found to be 1.4.

As a second attempt, $\bar{B}^{n+1}$ was calculated by subtracting (4.18) from (4.19) leaving the calculation for $\bar{A}$ as before; this did not show good results either. In searching for better results, a predictor-corrector technique was set up. The prediction is given by the equations below

$$(C^n + D^n)\,\bar{A}^* = \bar{G}^n + \bar{H}^n$$

and

$$D^n \frac{\bar{B}^*}{\Delta t} = E^n \bar{A}^* + \frac{\bar{B}^n}{\Delta t} - \bar{H}^n$$

with $\bar{B}^*$ the values of $C^{n+\frac{1}{2}}$, $D^{n+\frac{1}{2}}$, $E^{n+\frac{1}{2}}$, $\bar{G}^{n+\frac{1}{2}}$ and $\bar{H}^{n+\frac{1}{2}}$ were calculated, and those values used for the corrector equation as follows

$$(C^{n+\frac{1}{2}} + E^{n+\frac{1}{2}})\,\bar{A}^{n+\frac{1}{2}} = \bar{G}^{n+\frac{1}{2}} + \bar{H}^{n+\frac{1}{2}}$$

and

$$D^{n+\frac{1}{2}} \frac{\bar{B}^{n+\frac{1}{2}}}{\Delta t} = E^{n+\frac{1}{2}} A^{n+\frac{1}{2}} + \frac{\bar{B}^n}{\Delta t} - \bar{H}^{n+\frac{1}{2}}$$

The same iterative scheme already mentioned was used to calculate $\bar{A}^{n+\frac{1}{2}}$

and $\bar{B}^{n+1}$. Again, the convergence criteria had to be very tight in order to obtain good results.

In all the SOR techniques, the number of iterations necessary to converge was a problem; round-off error was accumulated from one calculation to the next and the solution attained did not satisfy the equations being solved. This led to the alternative of using a Gauss-Jordan solution already mentioned and in all the results presented, this method has been used.

Graphs of saturation versus distance for different water injection rates are shown in figures 4, 5, 6, and 7. The cumulative oil produced, as compared with the Buckley-Leverett method is given in tables 2, 3, 4 and 5. A difficulty was encountered in fitting the sharp front as the one obtained by the Buckley-Leverett [10] solution.

The influence of Δx in the simulation is shown in figure 8, and we can see that for small increases in Δx, the closer the front is to the position found by Buckley-Leverett. Jennings[11] has shown the order of approximation is about Δx^4. The time step is an important factor for this simulator and the magnitude of Δt depends on the accuracy desired. Further, there appears to be an important interaction between Δt and Δx. The water saturations obtained show a certain oscillation between values going from below the irreducible saturation to values above the one corresponding to the residual oil saturation. This oscillation seems to be the cause for the water front to be behind the position obtained in the Buckley-Leverett solution. As the time step is decreased, the oscillations also decrease, the front moves faster and the time to break through approaches the Buckley-Leverett Calculation.

Despite this fact the cummulative oil produced obtained by both techniques is essentially the same since in Galerkin's method the oil produced is calculated by obtaining the amount of flow at a point near the outlet of the system. The simulation is stopped when breakthrough is attained.

To have an idea of the running times used we have the following results. For a 50 feet long model with a cross sectional area of one square foot and the rest of the parameter as specified before, a pore volume of 1.185 barrels results, and for a water injection rate of 0.1 BPD the breakthrough is at about 11.58 days. If the model is divided into 10 intervals of 5 feet, the running time to breakthrough is 1.18 minutes for Δt = 0.1 days. When Δt is decreased to 0.01 the running time goes to 6.25 minutes. The amount of core is rather small, about 9K.

RESERVOIR PARAMETERS

The following reservoir parameters were chosen, though any consistent set of data may be used.

$S_{wi} = 0.2$

$\phi = 0.2$

$A = 1.0 \text{ ft}^2$

$S_{or} = 0.15$

$\mu_o = 1.0 \text{ cp}$

$\mu_w = 1.0 \text{ cp}$

$k = 0.01 \text{ darcys}$

TABLE 1

Relative Permeability Data

S(fraction)	k_{rw} (fraction)	k_{ro} (fraction)
0.2	.000	.930
0.25	.003	.752
0.30	.013	.597
0.35	.028	.462
0.40	.049	.355
0.45	.079	.275
0.50	.117	.214
0.55	.164	.162
0.60	.220	.118
0.65	.285	.080
0.70	.357	.047
0.75	.432	.023
0.80	.513	.006
0.85	.600	.000

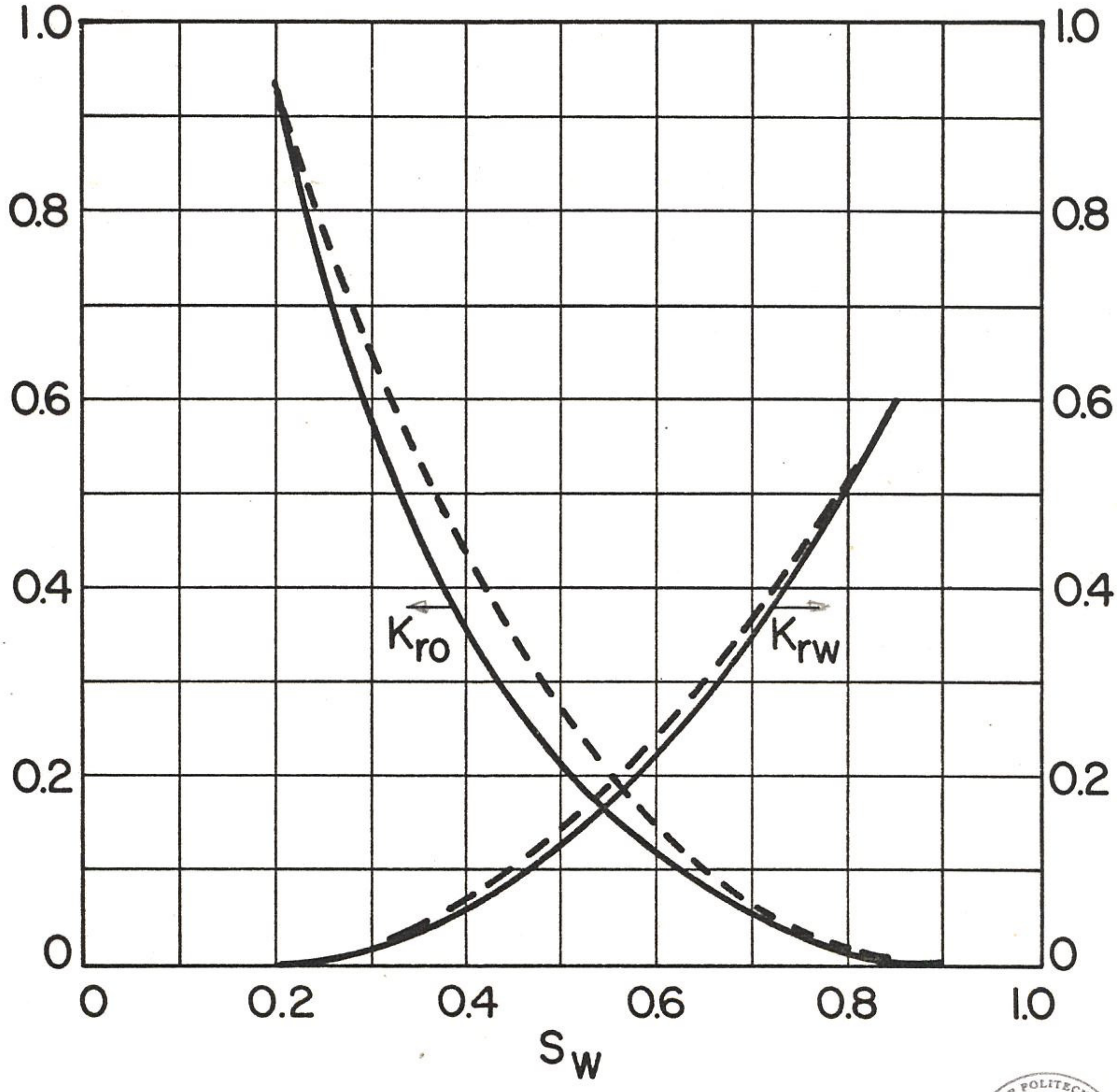


FIGURE 2
RELATIVE PERMEABILITY
CURVES

——— DATA
– – – POLYNOMIAL APPROXIMATION

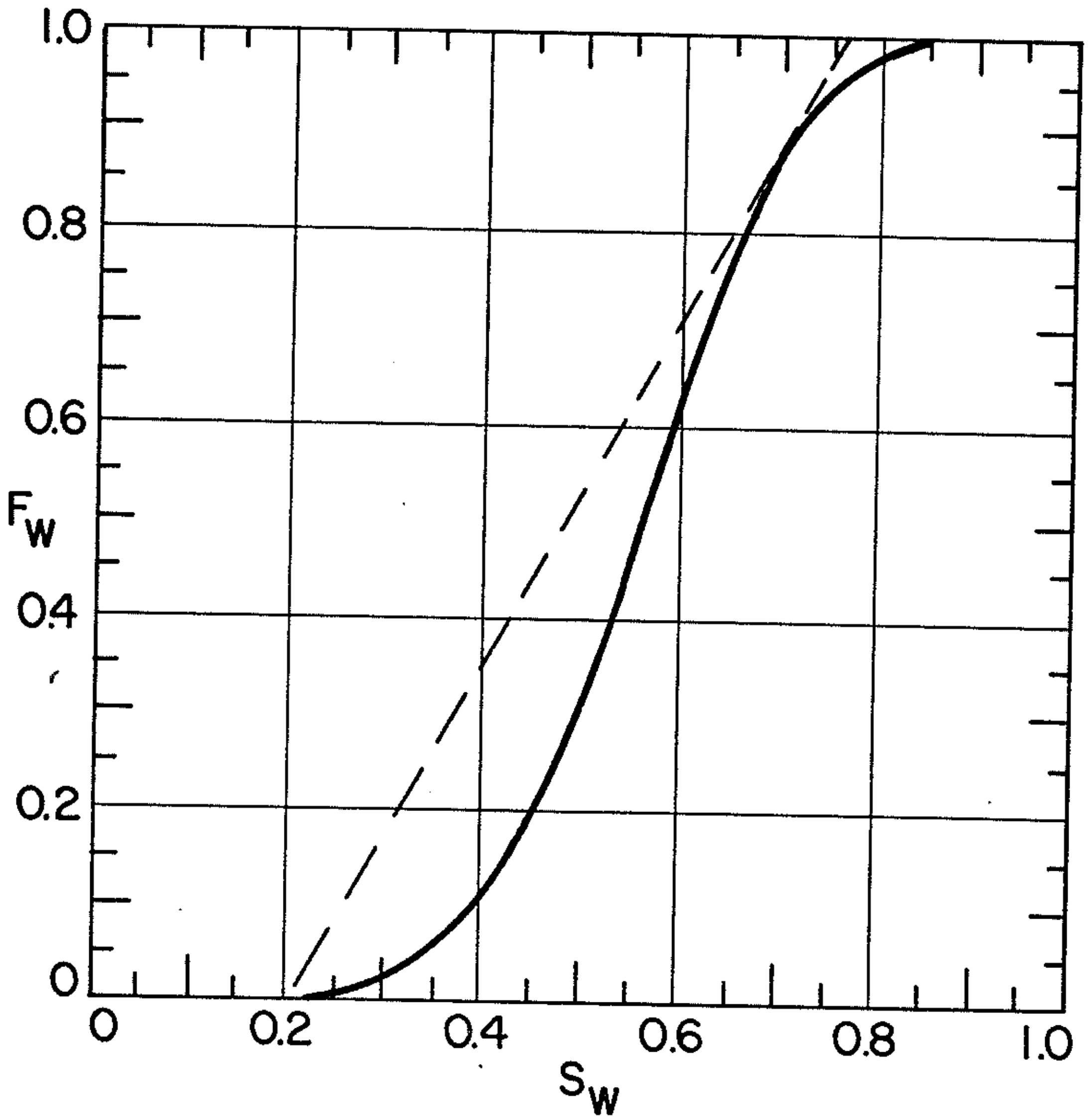


FIGURE 3

FRACTIONAL FLOW OF WATER

TABLE 2

Q_{IN} = 1 BPD

Time (days)	Cummulative Oil Produced (BPD) Buckley-Leverett	Galerkin
0	0	0
20.3	20.3	20.3
40.3	40.3	40.288
60.3	60.3	60.354
80.3	80.3	79.941
100.3	100.3	97.835

Area = 10 ft^2

L = 500 ft

t_B = 101.5 days (Buckley-Leverett)

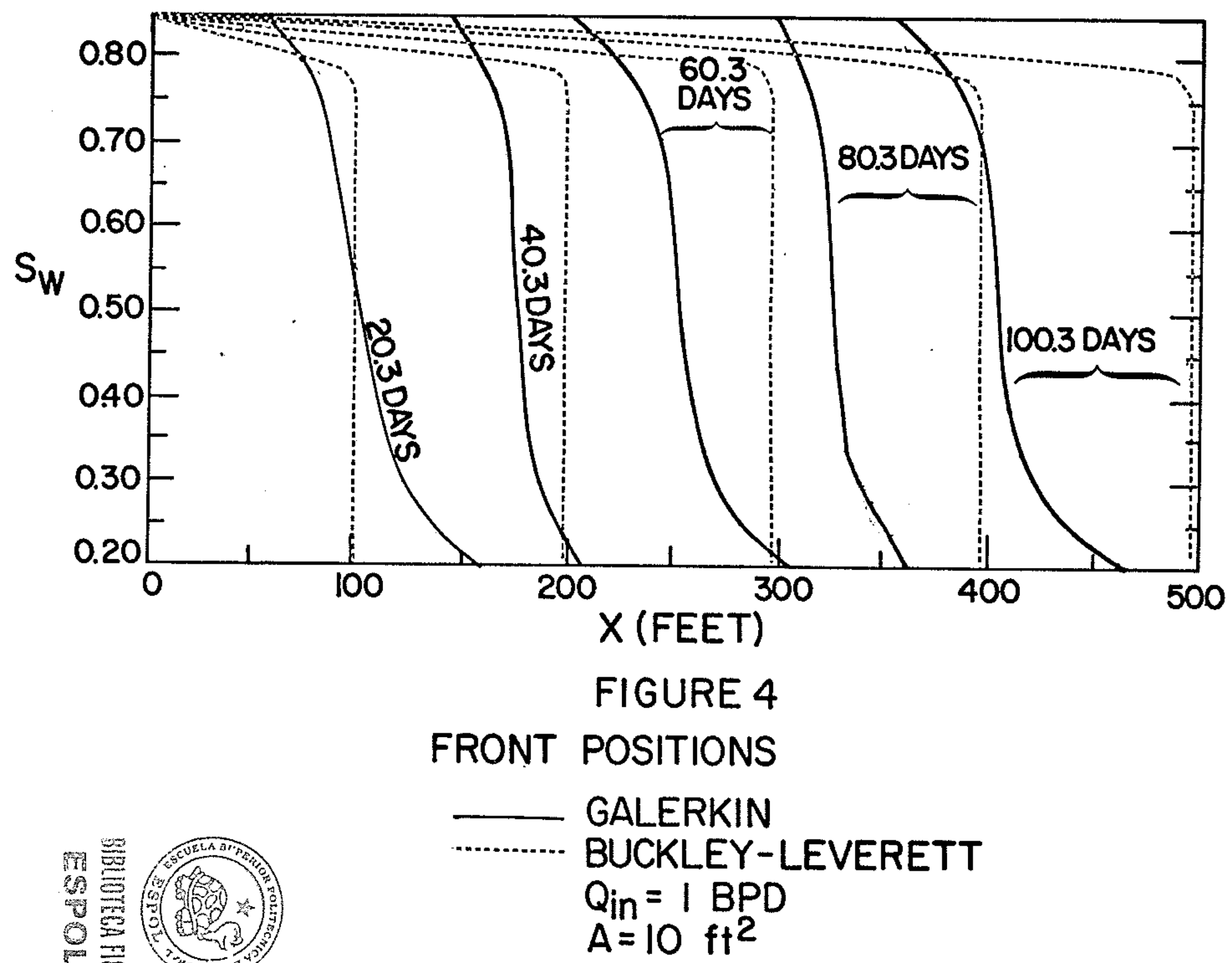


FIGURE 4

FRONT POSITIONS

TABLE 3

$Q_{IN} = 2$ BPD

Time (days)	Cummulative Oil Produced (BPD) Buckley-Leverett	Galerkin
0	0	0
9.9	19.8	19.801
20.3	40.6	40.588
29.9	59.8	59.832
40.3	80.6	80.302
49.9	99.8	98.513

Area = 10 ft^2

L = 500 ft

t_B = 50.7 days (Buckley-Leverett)

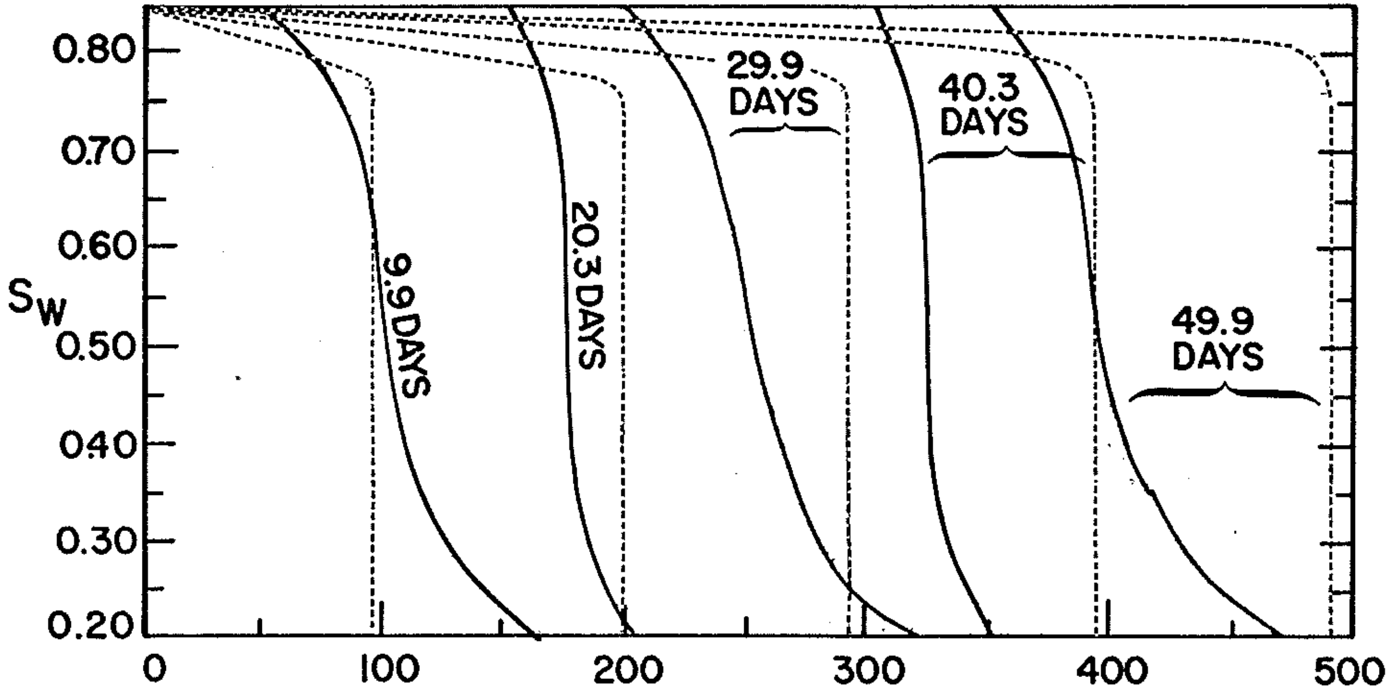


FIGURE 5
FRONT POSITIONS
——— GALERKIN
-------- BUCKLEY-LEVERETT
Q_{in} = 2 BPD
A = 10 ft^2

TABLE 4

Q_{IN} = 3 BPD

Time (days)	Cummulative Oil Produced (BPD) Buckley-Leverett	Galerkin
0	0	0
5.9	17.7	17.7
11.9	35.7	35.706
19.9	59.7	59.705
25.9	77.7	77.84
33.9	101.7	100.098

Area = 10 ft^2

L = 500 ft

t_B = 33.8 days (Buckley_Leverett)

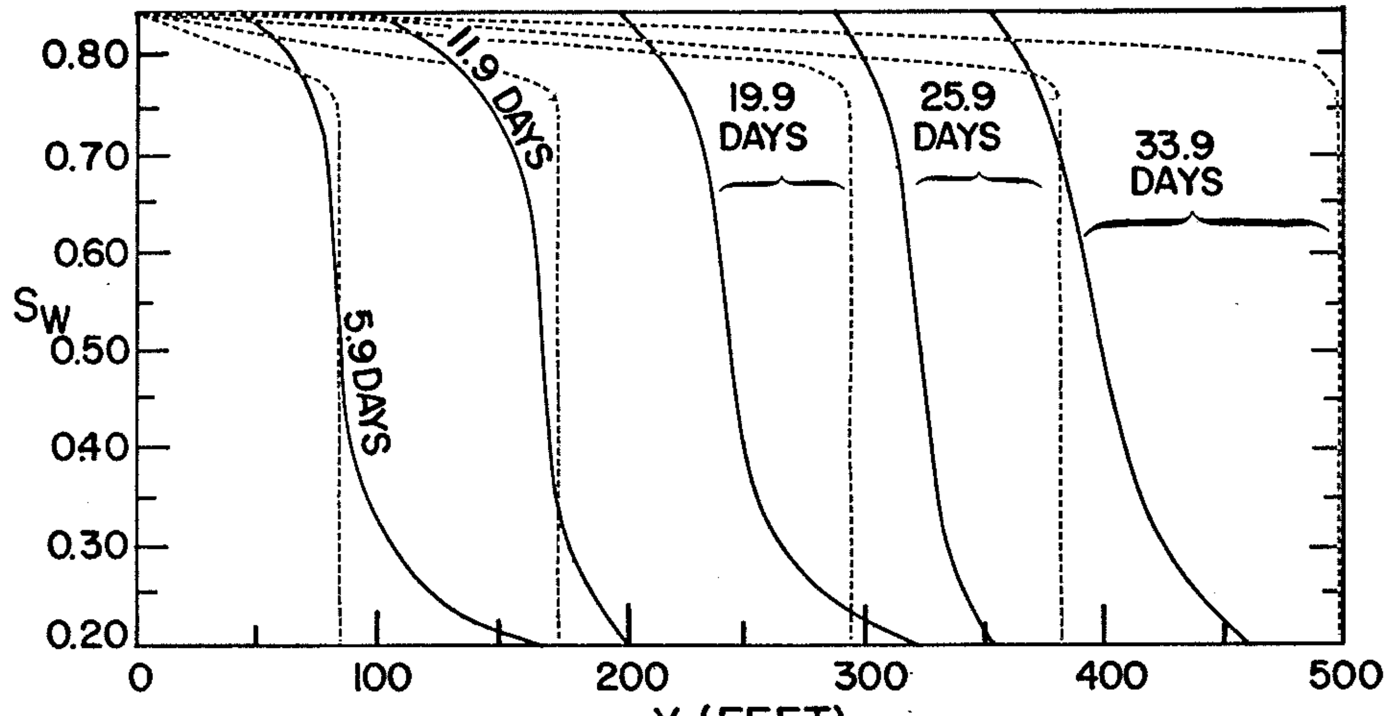


FIGURE 6

FRONT POSITIONS

——— GALERKIN
------- BUCKLEY-LEVERETT
Q_{in} = 3 BPD
A = 10 ft^2

Julio 18 de 1979

Ingeniero
Eduardo Rivadeneira, VICE-RECTOR
ESCUELA SUPERIOR POLITECNICA
Guayaquil

Pongo a vuestra consideración la revalidación del título de "Master of Science in Petroleum Engineering" que me otorgara la Universidad de Wyoming de los Estados Unidos de Norteamérica. Solicito a Ud. y por su intermedio al organismo correspondiente que éste título sea revalidado con el de "INGENIERO DE PETROLEOS" que otorga la ESPOL, para lo cual, me acojo al Artículo 10 del Reglamento para la revalidación e/o inscripción de títulos académicos y profesionales obtenidos en el extranjero, ya que me encuentro prestando servicios en esta Institución, ininterrumpidamente desde el 19 de Septiembre de 1974.

Acompaño a ésta solicitud la traducción legal del título, realizada por el juzgado noveno provincial del Guayas, un certificado de las materias aprobadas en la Universidad de Wyoming, copia del Comprobante Militar y Certificado de prestar servicios en esta Institución desde la fecha antes indicada.

Atentamente,

Ing. Carlos J. Arnao R.
PROFESOR DEL DPTO. ING.
GEOLOGIA, MINAS Y PETROLEO

CA/cpb

TABLE 5

Q_{IN} = 4 BPD

Time (days)	Cummulative Oil Produced (BPD) Buckley_Leverett	Galerkin
0	0	0
5.1	20.4	20.401
9.9	39.6	39.596
15.1	60.4	60.396
19.9	80.6	79.705
25.1	100.4	99.297

Area = 10 ft^2

L = 500 ft

t_B = 25.4 days (Buckley-Leverett)

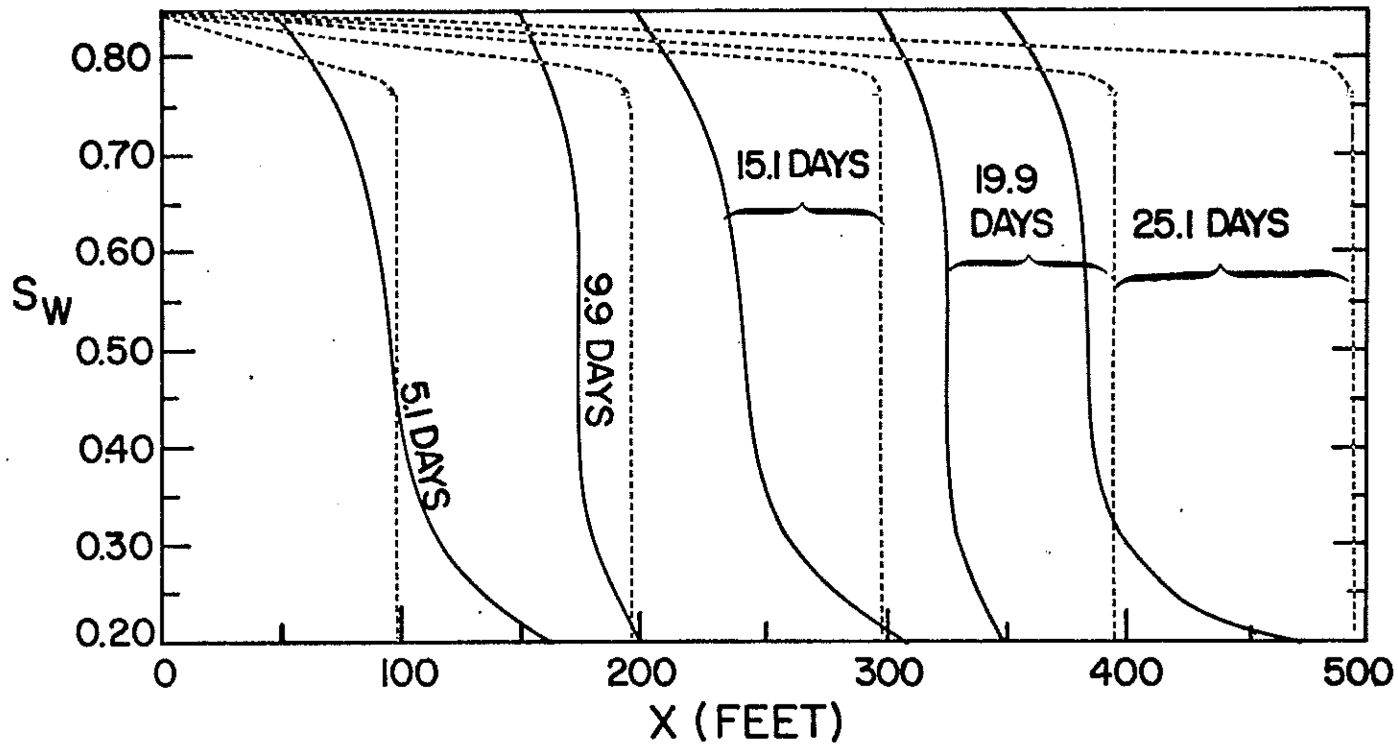


FIGURE 7

FRONT POSITIONS

——— GALERKIN
- - - - - BUCKLEY-LEVERETT

Q_{in} = 4 BPD
A = 10 ft^2

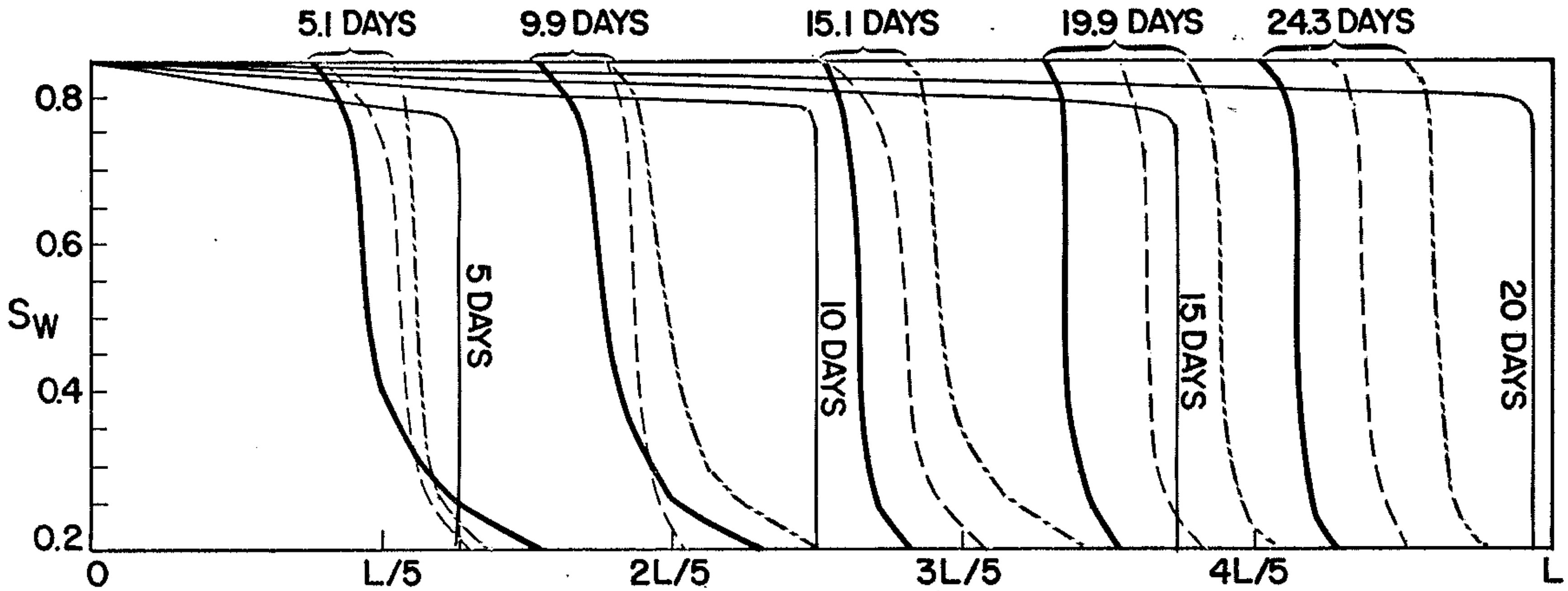
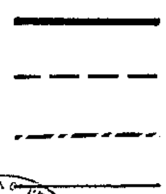


FIGURE 8



ΔT = 0.1 days
ΔT = 0.05 days
ΔT = 0.01 days
BUCKLEY-LEVERETT

Q_{in} = 0.1 BPD
A= 1 ft^2
ΔX = L/20

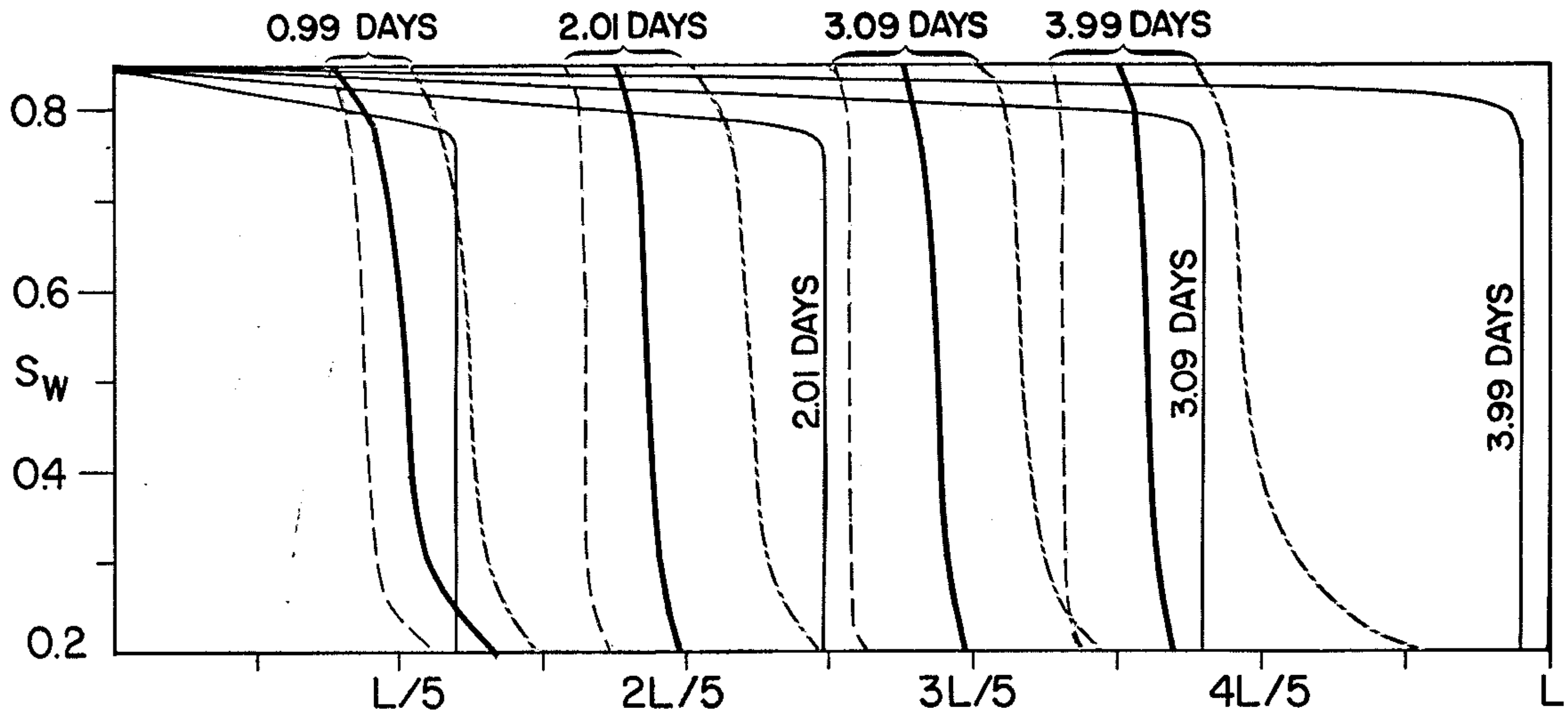


FIGURE 9

- - - - $\Delta X = L/40$
━━━ $\Delta X = L/20$
-·-·- $\Delta X = L/10$
——— Buckley-Leverett

$Q_{in} = 0.1$ BPD
$A = 1\ ft^2$
$\Delta T = 0.01$ days

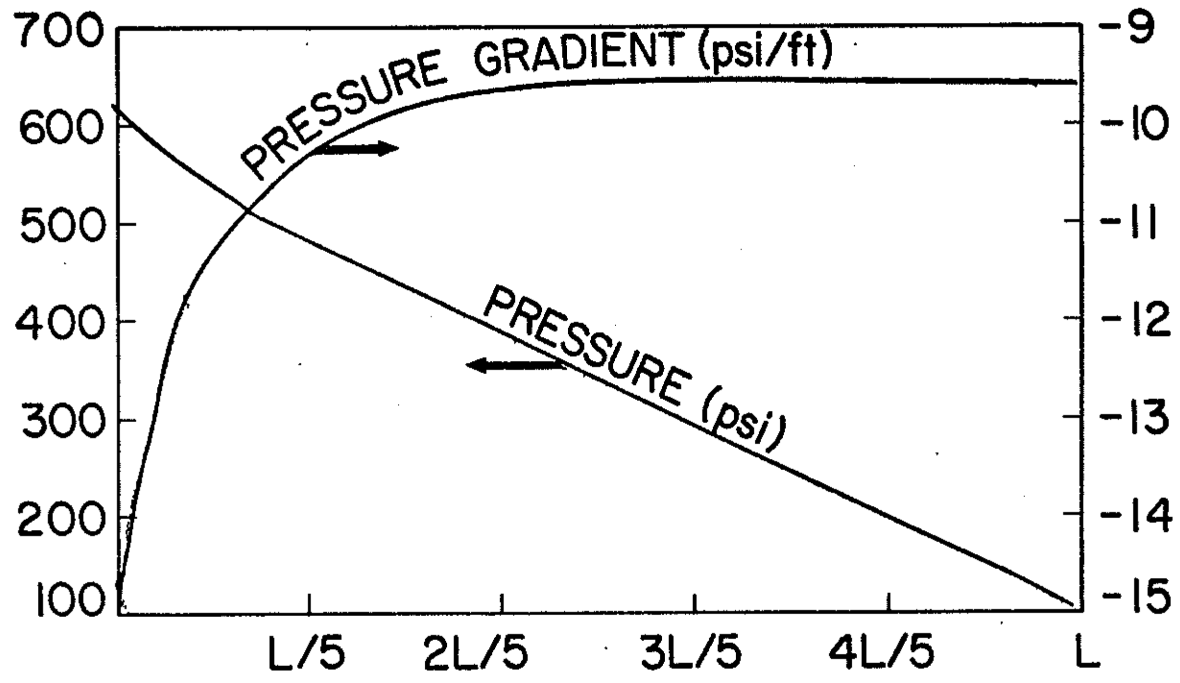


FIGURE 10

Q_{in} = 0.1 BPD
A = 1 ft^2
L = 50 ft
ΔX = 5 ft
Δt = 0.01 days
t = 0.16 days

CONCLUSIONS

1. Galerkin's method may have some utility in the solution of the problem of two-phase, one-dimensional flow in a porus media.
2. It was apparent in this work that the backward time differencing technique used was inadequate. A higher order time differencing scheme would allow larger and more realistic time step size.
3. In this study, the Galerkin solution resulted in a production history which was similar to that calculated by Buckley-Leverett. However, the saturation distribution in the model was different.
4. There appeared to be an interaction between the size of the time step and the size of the space increment in this study.
5. It is possible to perform all the integrations required on the Galerkin's solution in closed form by using a polynomial approximation for relative permeabilities. As a result, time consuming quadrature schemes are not required.

APPENDIX A

Notation

A	a vector
B	a vector
C	coefficient matrix
D	coefficient matrix
E	coefficient matrix
F	coefficient matrix
f	dimensionless function of basis functions
G	a vector
H	a vector
k	absolute permeability darcy units. Also used as a subscript.
k_r	relative permeability, fraction
k_{ro}	relative permeability to oil, fraction
k_{rw}	relative permeability to water, fraction
L	length, feet
L_1	linear operator
L_2	linear operator
P	pressure, psi
P_i	initial pressure, psi
P*	approximation to pressure, psi
Q_{IN}	water injection rate, BPD
S	water saturation, fraction
S_o	oil saturation, fraction

S_{or} residual oil saturation, fraction

S_w water saturation, fraction

S_{wi} irreducible water saturation, fraction

S^* approximation to water saturation, fraction

t time, days

t_B time to breakthrough, days

v velocity, barrels per day per square foot

w basis function

x distance, feet

y dimension less distance

z relative permeability coefficient

ϕ porosity, fraction

ρ denisty gradient, psi per foot

μ viscosity, centipoises

μ_o oil viscosity, centipoises

μ_w water viscosity, centipoises

Δ finite difference operator

∇ nabla operator

$-$ denotes a vector quality

APPENDIX B

CUBIC SMOOTH BASIC FUNCTIONS

The equations for this type of function are:

$$w_i^{[1]}(x) = \begin{cases} \dfrac{(-2x + 3x_i - x_{i-1})(x-x_{i-1})^2}{(x_i - x_{i-1})^3}, & x_{i-1} \le x \le x_i \\[2ex] \dfrac{(2x - 3x_i + x_{i+1})(x - x_{i+1})^2}{(x_{i+1} - x_i)^3}, & x_i \le x \le x_{i+1} \end{cases}$$

$$w_i^{[2]}(x) = \begin{cases} \dfrac{(x - x_{i-1})^2 (x - x_{i-1})}{(x_i - x_{i-1})^2}, & x_{i-1} \le x \le x_i \\[2ex] \dfrac{(x - x_{i+1})^2 (x - x_i)}{(x_i - x_{i+1})^2}, & x_i \le x \le x_{i+1} \end{cases}$$

$w_i^{[1]}(x)$ has a slope of zero at x_{i-1}, x_i and x_{i+1}, the amplitude at x_i is one. $w_i^{[2]}(x)$ has slope zero at x_{i-1} and x_{i+1}, while it is 1 at x_i.

$w_i^{[1]}(x)$ is the amplitude basic function and is associated with Pressure and Saturation while $w_i^{[2]}(x)$ is the gradient basic function and associates with $\frac{dP}{dx}$ and $\frac{dS}{dx}$.

The above mentioned functions are much easier to handle in dimensionless form. Let

$$y = \frac{x - x_{i-1}}{h}, \quad x_{i-1} \le x \le x_i$$

and

$$y = \frac{x - x_i}{h}, \quad x_i \leq x \leq x_{i+1}$$

where y takes values between zero and one and dx = h dy. For the sake of simplicity, let

$$w_i^{[1]}(x) = f_1$$

$$w_i^{[2]}(x) = f_2, \quad x_i \leq x \leq x_{i+1}$$

$$w_i^{[1]}(x) = f_3$$

$$w_i^{[2]}(x) = f_4, \quad x_{i-1} \leq x \leq x_i$$

With this new nomenclature, the cubic basic functions become

$$f_1 = 2y^3 - 3y^2 + 1$$

$$f_2 = h(y^3 - 2y^2 + y)$$

$$f_3 = -2y^3 + 3y^2$$

$$f_4 = h(y^3 - y^2)$$

and their derivatives are:

$$f_1' = \frac{1}{h}(6y^2 - 6y)$$

$$f_2' = 3y^2 - 4y + 1$$

$$f_3' = -\frac{1}{h}(6y^2 - 6y)$$

$$f_4' = 3y^2 - 2y$$

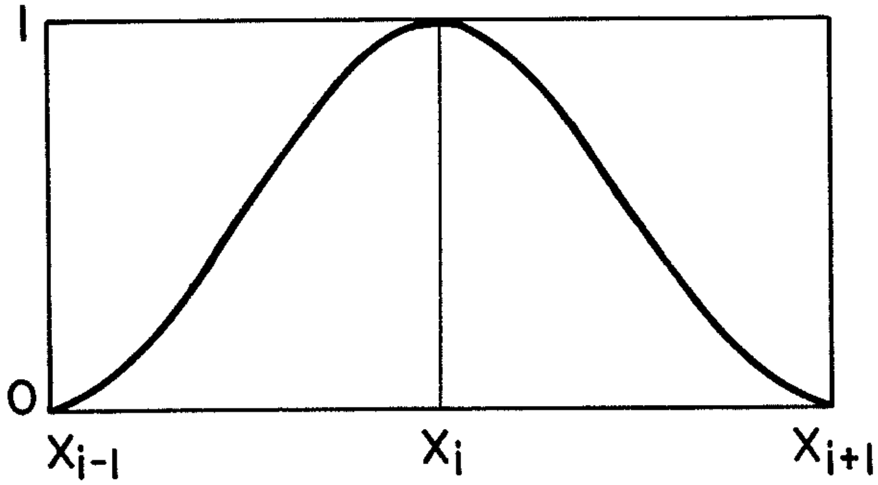


FIGURE 11
FUNCTION $\omega_i^{[1]}$

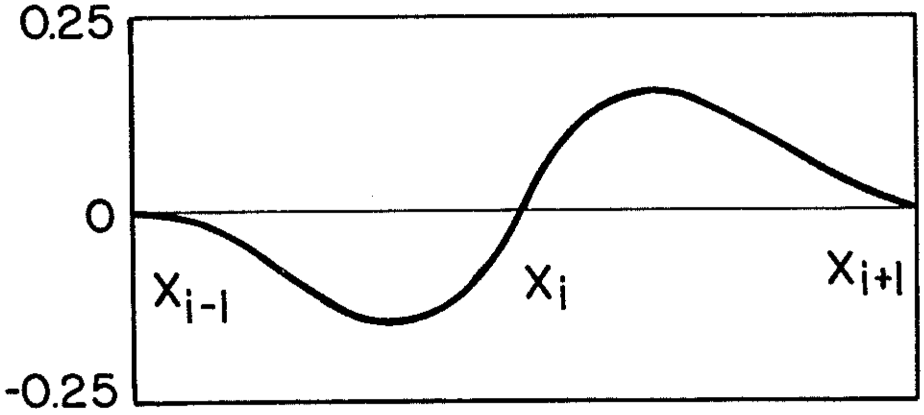


FIGURE 12
FUNCTION $\omega_i^{[2]}$

APPENDIX C

This appendix shows the elements in the i^{th} row in (2.16) for the matrices C, E, D and F. The first subscript refers to the row and the second one to the column.

Matrix C:

$$L_{1,1} = -\int_{x_{i-1}}^{x_i} M\, f_3' f_1'\, dx \qquad L_{1,2} = -\int_{x_{i-1}}^{x_i} M\, f_3' f_2'\, dx$$

$$L_{2,1} = -\int_{x_{i-1}}^{x_i} M\, f_4' f_1'\, dx \qquad L_{2,2} = -\int_{x_{i-1}}^{x_i} M\, f_4' f_2'\, dx$$

$$J_{1,1} = -\int_{x_{i-1}}^{x_i} M\, f_3' f_3'\, dx - \int_{x_i}^{x_{i+1}} M\, f_1' f_1'\, dx$$

$$J_{1,2} = -\int_{x_{i-1}}^{x_i} M\, f_3' f_4'\, dx - \int_{x_i}^{x_{i+1}} M\, f_1' f_2'\, dx$$

$$J_{2,1} = -\int_{x_{i-1}}^{x_i} M\, f_4' f_3'\, dx - \int_{x_i}^{x_{i+1}} M\, f_2' f_1'\, dx$$

$$J_{2,2} = -\int_{x_{i-1}}^{x_i} M\, f_4' f_4'\, dx - \int_{x_i}^{x_{i+1}} M\, f_2' f_2'\, dx$$

$$R_{1,1} = -\int_{x_i}^{x_{i+1}} M\, f_1' f_3'\, dx \qquad R_{12} = -\int_{x_i}^{x_{i+1}} M\, f_1' f_4'\, dx$$

$$R_{21} = - \int_{x_i}^{x_{i+1}} M\, f_2' f_3' \, dx \qquad R_{2,2} = - \int_{x_i}^{x_{i=1}} M\, f_2' f_4' \, dx$$

Matrix E:

For this matrix all the M's in the equation above are substituted by N's.

Matrices D and F:

$$L_{1,1} = \frac{\phi}{6.328} \int_{x_{i-1}}^{x_i} f_3 f_1 \, dx \qquad L_{1,2} = \frac{\phi}{6.328} \int_{x_{i-1}}^{x_i} f_3 f_2 \, dx$$

$$L_{2,1} = \frac{\phi}{6.328} \int_{x_{i-1}}^{x_i} f_4 f_1 \, dx \qquad L_{2,2} = \frac{\phi}{6.328} \int_{x_{i-1}}^{x_i} f_4 f_2 \, dx$$

$$J_{1,1} = \frac{\phi}{6.328} \int_{x_{i-1}}^{x_i} f_3 f_3 \, dx + \frac{\phi}{6.328} \int_{x_i}^{x_{i+1}} f_1 f_1 \, dx$$

$$J_{1,2} = \frac{\phi}{6.328} \int_{x_{i-1}}^{x_i} f_3 f_4 \, dx + \frac{\phi}{6.328} \int_{x_i}^{x_{i+1}} f_1 f_2 \, dx$$

$$J_{2,1} = \frac{\phi}{6.328} \int_{x_{i-1}}^{x_i} f_4 f_3 \, dx + \frac{\phi}{6.328} \int_{x_i}^{x_{i+1}} f_2 f_1 \, dx$$

$$J_{2;2} = \frac{\phi}{6.328} \int_{x_{i-1}}^{x_i} f_4 f_4 \, dx + \frac{\phi}{6.328} \int_{x_i}^{x_{i+1}} f_2 f_2 \, dx$$

$$R_{11} = \frac{\phi}{6.328} \int_{x_i}^{x_{i+1}} f_1 f_3 \, dx \qquad R_{12} = \frac{\phi}{6.328} \int_{x_i}^{x_{i+1}} f_1 f_4 \, dx$$

$$R_{21} = \frac{\phi}{6.328}\int_{x_i}^{x_{i+1}} f_2 f_3 \, dx \qquad R_{22} = \frac{\phi}{6.328}\int_{x_i}^{x_{i+1}} f_2 f_4 \, dx$$

When the integration is carried out the following results are obtained.

$$L = \begin{vmatrix} \frac{9}{70}h & \frac{13}{420}h^2 \\ -\frac{13}{420}h^2 & -\frac{h^3}{140} \end{vmatrix}$$

$$J = \begin{vmatrix} \frac{26}{35}h & 0 \\ 0 & \frac{2}{105}h^3 \end{vmatrix}$$

$$R = \begin{vmatrix} \frac{9}{70}h & -\frac{13}{420}h^2 \\ \frac{13}{420}h^2 & -\frac{h^3}{140} \end{vmatrix}$$

APPENDIX D

The Computer Program

The computer program was written in FORTRAN and consists of the main program and five subroutines: FIRINT, SECINT, POLY, PERM and SOLVER.

The main program dimensions all the arrays, reads the data in, sets the relative permeability curve coefficients, executes the loop for every time step and prints out the results. Subroutine PERM calculates the coefficients of the polynomial

$$k_r = k_7 + k_1 y + k_2 y^2 + k_3 y^3 + k_4 y^4 + k_5 y^5 + k_6 y^6$$

to express the relative permeabilities as a function of distance. k_r may be k_{ro} or k_{rw}, depending on how the subroutine is called.

Subroutine POLY calculates the integrals of the form

$$\int M \, w'_m \, w'_n \, dx$$

and

$$\int N \, w'_m \, w'_n \, dx$$

pertaining to the elements of matrices C and E and subroutine FIRINT sets those values to the corresponding elements. Subroutine SECINT calculates the entries of matrices D and F and subroutine SOLVER solves the system of equations to calculate the vectors $\overline{A}$ and $\overline{B}$.

MAIN PROGRAM

```
      REAL K,KRW1,KRWN,KRON,KRO1
      COMMON /GROUP1/K(7)
      COMMON /GROUP2/Y1,Y2,Y3,Y4,Y5,Y6
      COMMON /GROUP3/AOLD(50),BOLD(50),ANEW(50),BNEW(50),ASAVER(50),
     1BSAVER(50)
      COMMON /GROUP4/C(50,6),D(50,6),E(50,6),F(50,6),Q(50,6),QM(5,50)
      COMMON /GROUP5/ZO(3),ZW(3),KRW1,KRWN,KRO1,KRON
      COMMON /GROUP6/EPSI,SWI,POR,SOR,AREA,OILVIS,WATVIS,QIN,ABPER,DT,
     1N,ITMAX
      COMMON /GROUP7/ITER,ITER1
      DIMENSION FLOW(50),QT(50)
C
      READ 3,POR,SOR,SWI,OILVIS,WATVIS,ABPER,N
      READ 3,EPSI,ITMAX
      READ 3,AREA,H
      READ 3,QIN,DT
      READ 3,W
      READ 3,NP1,NPT,ISTART
    3 FORMAT (10F)
      OUTPUT EPSI,ITMAX,SWI,SOR,POR,AREA,H,OILVIS,WATVIS,QIN,ABPER,N,DT
      OUTPUT W,NP1,NPT
C
      ZO(1)=1.590355029585798
      ZO(2)=-3.742011834319525
      ZO(3)=2.201183431952662
      ZW(1)=5.680473372781064E=-2
      ZW(2)=-0.5680473372781064
      ZW(3)=1.420118343195266
      NM1=N-1
C
      IF (ISTART.EQ.1)GOTO 11
C     INITIAL CONDITIONS
      DO 10 I=1,NM1,2
```

```
      AOLD(I)=100.0
      AOLD(I+1)=0.0
      BOLD(I)=SWI
   10 BOLD(I+1)=0.0
      BOLD(1)=1.0-SOR
      NPIT=1
      NP=NP1
      NPT1=NPT+1
      TIME=0.0
      GOTO 12
   11 READ (3) NP,NPIT,TIME
      READ (3) (AOLD(I),I=1,50),(BOLD(I),I=1,50)
      REWIND (3)
      NPT1=NPIT+NPT
C
   12 CO 73 I=1,N
      ANEW(I)=AOLD(I)
   73 BNEW(I)=BOLD(I)
C     SECOND INTEGRAL MATRICES
      CALL SECINT(H)
C
      QTOT=0.
C
      DO 99 IT=NPIT,NPT1
      TIME =TIME+DT
      CALL FIRINT(H)
      CALL SOLVER(W)
      DO 50 I=1,N-1,2
      WPER=ZW(1)+ZW(2)*BNEW(I)+ZW(3)*BNEW(I)**2
      OPER=ZO(1)+ZO(2)*BNEW(I)+ZO(3)*BNEW(I)**2
      FLOW(I+1)=-1.127*ABPER*WPER*AREA*ANEW(I+1)/WATVIS
      FLOW(I)=-1.127*ABPER*OPER*AREA*ANEW(I+1)/OILVIS
   50 QT(I)=FLOW(I)+FLOW(I+1)
      QTOT=QTOT+QT(N-5)*DT
      IF (IT.NE.NP)GOTO 98
```

```
      PRINT 25,DT,TIME
   25 FORMAT (/,/,'DT=',E11.4,3X,'TIME=',E11.4,X,'DAYS')
      PRINT 83
   83 FORMAT (/,/,8X,'PRESSURE',5X,'PRES. GRADIENT',5X,'SATURATION',5X,
     1'SAT. GRADIENT',5X,'OIL FLOW',7X,'WATER FLOW',6X,'TOTAL FLOW')
      DO 84 I=1,NM1,2
   84 PRINT 67,I,ANEW(I),ANEW(I+1),BNEW(I),BNEW(I+1),FLOW(I),FLOW(I+1),
     1QT(I)
   67 FORMAT (I3,7E16.7)
      PRINT 68,QTOT
   68 FORMAT (/,'CUMMULATIVE FLOW RATE = ',F7.3,X,'BARRELS')
      NP=NP+NP1
   98 DO 87 I=1,N
      AOLD(I)=ANEW(I)
   87 BOLD(I)=BNEW(I)
   99 CONTINUE
      OUTPUT IT
      NPIT=NPT1+1
      WRITE (3) NP,NPIT,TIME
      WRITE (3) (ANEW(I),I=1,50),(BNEW(I),I=1,50)
      STOP
      END
```

```
      SUBROUTINE SOLVER(W)
      REAL K,KRW1,KRWN,KRON,KRO1
      COMMON /GROUP3/AOLD(50),BOLD(50),ANEW(50),BNEW(50),ASAVER(50),
     1BSAVER(50)
      COMMON /GROUP4/C(50,6),D(50,6),E(50,6),F(50,6),Q(50,6),QM(5,50)
      COMMON /GROUP5/ZO(3),ZW(3),KRW1,KRWN,KRO1,KRON
      COMMON /GROUP6/EPSI,SWI,POR,SOR,AREA,OILVIS,WATVIS,QIN,ABPER,DT,
     1N,ITMAX
      COMMON /GROUP7/ITER,ITER1
      DIMENSION G(50)
C
      DO 50 I=1,N
      DO 50 J=1,6
      D(I,J)=C(I,J)+E(I,J)
      E(I,J)=E(I,J)-C(I,J)
      C(I,J)=D(I,J)
   50 CONTINUE
      ANEW(N)=-QIN/(1.127*ABPER*AREA*(KRON/IOLVIS+KRWN/WATVIS))
      ANEW(2)=-QIN/(1.127*ABPER*AREA*(KRO1/OILVIS+KRW1/WATVIS))
      ANEW(N-1)=AOLD(N-1)
      NM1=N-1
      NM2=N-2
      NM3=N-3
      NM4=N-4
      G(1)=-C(1,4)*ANEW(2)
      G(3)=-C(3,2)*ANEW(2)
      G(4)=-C(4,2)*ANEW(2)
      DO 22 I=5,NM4
   22 G(I)=0,
      G(NM3)=-C(NM3,5)*ANEW(NM1)
      G(NM2)=-C(NM2,5)*ANEW(NM1)
      G(N)=-C(N,3)*ANEW(NM1)
      DO 23 I=4,NM2,2
      XM=C(I-1,1)/C(I,1)
      G(I)=G(I)*XM-G(I-1)
```

```
      DO 23 J=1,6
23    C(I,J)=C(I,J)*XM-C(I-1,J)
      DO 24 I=5,NM3,2
      XM=C(I-1,3)/C(I,1)
      G(I)=G(I)*XM-G(I-1)
      DO 25 J=1,4
25    C(I,J)=C(I,J)*XM-C(I-1,J+2)
      DO 31 J=5,6
31    C(I,J)=C(I,J)*XM
      XM=C(I,2)/C(I+1,2)
      G(I+1)=G(I+1)*XM-G(I)
      DO 24 J=2,6
24    (C(I+1,J)=C(I+1,J)*XM-C(I,J)
      XM=C(NM2,3)/C(N,1)
      G(N)=G(N)*XM-G(NM2)
      DO 34 J=1,4
34    C(N,J)=C(N,J)*XM-C(NM2,J+2)
      XM=C(1,3)/C(3,1)
      G(3)=G(3)*XM-G(1)
      C(3,1)=C(3,1)*XM-C(1,3)
      DO 26 J=3,4
26    C(3,J)=C(3,J)*XM-C(1,J+2)
      DO 32 J=5,6
32    C(3,J)=C(3,J)*XM
      XM=C(3,3)/C(4,3)
      G(4)=G(4)*XM-G(3)
      DO 27 J=3,6
27    C(4,J)=C(4,J)*XM-C(3,J)
      DO 28 I=5,NM3,2
      XM=C(I-1,4)/C(I,2)
      G(I)=G(I)*XM-G(I-1)
      DO 29 J=2,4
29    C(I,J)=C(I,J)IIXM-C(I-1,J+2)
      DO 33 J=5,6
33    C(I,J)=C(I,J)*XM
```

```
      XM=C(I,3)/C(I+1,3)
      G(I+1)=G(I+1)*XM-G(I)
      DO 28 J=3,6
   28 C(I+1,J)=C(I+1,J)*XM-C(I,J)
      XM=(C(NM2,4)/C(N,2)
      G(N)=G(N)*XM-G(NM2)
      DO 35 J=2,4
   35 C(N,J)=C(N,J)*XM-C(NM2,J+2)
      ANEW(N)=G(N)/C(N,4)
      ANEW(NM2)=(G(NM2)-C(NM2,6)*ANEW(N))/C(NM2,4)
      ANEW(NM3)=(G(NM3)-C(NM3,6)*ANEW(N)-C(NM3,4)*ANEW(NM2))/C(NM3,3)
      DO 30 L=4,NM4,2
      I=NM4-(L-4)
      ANEW(I)=(G(I)-C(I,6)*ANEW(I+2)-C(I,5)*ANEW(I+1))/C(I,4)
   30 ANEW(I-1)=(G(I-1)-C(I-1,6)*ANEW(I+2)-C(I-1,5)*ANEW(I+1)
     1    -C(I-1,4)*ANEW(I))/C(I-1,3)
      ANEW(1)=(G(1)-C(1,6)*ANEW(4)-C(1,5)*ANEW(3))/C(1,3)
C
C     CALCULATE B'S
      DO 70 I=3,NM3,2
      G(I)=E(I,1)*ANEW(I-2)+E(I,2)*ANEW(I-1)+E(I,3)*ANEW(I)
     1+E(I,4)*ANEW(I+1)+E(I,5)*ANEW(I+2)+E(I,6)*ANEW(I+3)
      I1=I+1
      G(I1)=E(I1,1)*ANEW(I-2)+E(I1,2)*ANEW(I-1)+E(I1,3)*
     1ANEW(I)+E(I1,4)*ANEW(I+1)+E(I1,5)*ANEW(I+2)+E(I1,6)*
     2ANEW(I+3)
      G(I)=G(I)+(F(I,1)*BOLD(I-2)+F(I,2)*BOLD(I-1)+F(I,3)*
     1BOLD(I)+F(1,4)*BOLD(I+1)+F(I,5)*BOLD(I+2)+F(I,6)*
     2BOLD(I+3))
   70 G(I1)=G(I1)+(F(I1,1)*BOLD(I-2)+F(I1,2)*BOLD(I-1)+F(I1,3)*
     1BOLD(I)+F(I1,4)*BOLD(I+1)+F(I1,5)*BOLD(I+2)+F(I1,6)*
     2BOLD(I+3))
      G(1)=E(1,3)*ANEW(1)+E(1,4)*ANEW(2)+E(1,5)*ANEW(3)
     1+E(L,6)*ANEW(4)+(F(1,3)*BOLD(1)+F(1,4)*BOLD(2)+F(1,5)*
     2BOLD(3)+F(1,6)*BOLD(4)
```

```
      G(2)=E(2,3)*ANEW(1)+E(2,4)*ANEW(2)+E(2,5)*ANEW(3)
     1+E(2,6)*ANEW(4)+(F(2,3)*BOLD(1)+F(2,4)*BOLD(2)+F(2.5)*
     2BOLD(3)+F(2,6)*BOLD(4))
      G(N-1)=E(N-1,1)*ANEW(N-3)+E(N-1,2)*ANEW(N-2)+E(N-1,3)*
     1ANEW(N-1)+E(N-1,4)*ANEW(N)+(F(N-1,1)*BOLD(N-3)+F(N-1,2)*
     2BOLD(N-2)+F(N-1,3)*BOLD(N-1)+F(N-1,4)*BOLD(N)N
      G(N)=E(N,1)*ANEW(N-3)+E(N,2)*ANEW(N-2)+E(N,3)*ANEW(N-1)
     1+E(N,4)*ANEW(N)+(F(N,1)*BOLD(N-3)+F(N,2)*BOLD(N-2)
     2+F(N,3)*BOLD(N-1)+F(N,4)*BOLD(N))
C
      G(3)=G(3)-F(3,1)*BNEW(1)-F(3,2)*BNEW(2)
      G(4)=G(4)-F(4,1)*BNEW(1)-F(4,2)*BNEW(2)
      DO 53 I=6,N,2
      XM=QM(1,I)
   53 G(I)=G(I)*XM-G(I-1)
      G(5)=G(5)*QM(2,5)-G(3)
      G(6)=G(6)*QM(3,5)-G(5)
      DO 54 I=7,NM1,2
      XM=QM(2,I)
      G(I)=G(I)*XM-G(I-1)
      XM=QM(3,I)
   54 G(I+1)=G(I+1)*XM-G(I)
      DO 58 I=5,NM1,2
      XM=QM(4,I)
      G(I)=G(I)*XM-G(I-1)
      XM=QM(5,I)
   58 G(I+1)=G(I+1)*XM-G(I)
      BNEW(N)=G(N)/Q(N,4)
      BNEW(NM1)=(G(NM1)-Q(NM1,4)*BNEW(N))/Q(NM1,3)
      DO 80 L=2,NM4,2
      I=NM4-(L-4)
      BNEW(I)=(G(I)-Q(I,6)*BNEW(I+2)-Q(I,5)*BNEW(I+1))/Q(I,4)
   80 BNEW(I-1)=(G(I-1)-Q(I-1,6)*BNEW(I+2)-Q(I-1,5)*BNEW(I+1)
     1      -Q(I-1,4)*BNEW(I))/Q(I-1,3)
C
```

```
      RETURN
      END
```

```
      SUBROUTINE SECINT(H)
      COMMON /GROUP4/C(50,6),D(50,6),E(50,6),F(50,6),Q(50,6),QM(5,50)
      COMMON /GROUP6/EPSI,SWI,POR,SOR,AREA,OILVIS,WATVIS,QIN,ABPER,DT,
     1N,ITMAX
      NM1=N-1
      NM2=N-2
      NM3=N-3
      X11=9.*H/70.
      X12=13.*(H**2)/420.
      X13=-(H**3)/140.
      X14=26.*H/35.
      X15=2.*(H**3)/105.
      DO 20 I=3,NM3,2
      F(I,1)=X11
      F(I,2)=X12
      F(I,3)=X14
      F(I,4)=0.
      F(I,5)=X11
      F(I,6)=X12
      I1=I+1
      F(I1,1)=-X12
      F(I1,2)=X13
      F(I1,3)=0.
      F(I1,4)=X15
      F(I1,5)=X12
   20 F(I1,6)=X13
      F(1,3)=13.0*H/35.0
      F(1,4)=11.*(H**2)/210.
      F(1,5)=X11
      F(1,6)=-X12
      F(2;3)=F(1,4)
      F(2,4)=(H**3)/105.
      F(2,5)=X12
      F(2,6)=X13
      F(N-1,2)=X12
```

```
      F(N-1,4)=-11.*(H**2)/210.
      F(N-1,3)=13.0*H/35.0
      F(N-1,1)=X11
      F(N-1)=-X12
      F(N,2)=X13
      F(N,3)=F(N-1,4)
      F(N,4)=F(2,4)
      F(NM1,6)=1.
      DO 22 I=1,N
      DO 22 J=1,6
      F(I,J)=F(I,J)*2.*POR/(6.328*DT)
22    Q(I,J)=F(I,J)
      DO 53 I=6,N,2
      XM=Q(I-1,1)/Q(I,1)
      QM(1,I)=XM
      DO 53 J=1,6
53    Q(I,J)=Q(I,J)*XM=Q(I-1,J)
      XM=Q(3,3)/Q(5,1)
      QM(2,5)=XM
      DO 64 J=1,4
64    Q(5,J)=Q(5,J)*XM-Q(3,J+2)
      DO 65 J=5,6
65    Q(5,J)=Q(5,J)*XM
      XM=Q(5,2)/Q(6,2)
      QM(3,5)=XM
      DO 66 J=2,6
66    Q(6,J)=Q(6,J)*XM-Q(5,J)
      DO 54 I=7,NM1,2
      XM=Q(I-1,3)/Q(I,1)
      QM(2,I)=XM
      DO 55 J=1,4
55    Q(I,J)=Q(I,J)*XM-Q(I-1,J+2)
      DO 61 J=5,6
61    Q(I,J)=Q(I,J)*XM
      XM=Q(I,2)/Q(I+1,2)
```

```
        QM(3,I)=XM
        DO 54 J=2,6
    54  Q(I+1,J)=Q(I+1,J)*XM-Q(I,J)
        DO 58 I=5,NM1,2
        XM=Q(I-1,4)/Q(I,2)
        QM(4,I)=XM
        DO 59 J=2,4
    59  Q(I,J)=Q(I,J)*XM-Q(I-1,J+2)
        DO 63 J=5,6
    63  Q(I,J)=Q(I,J)*XM
        XM=Q(I,3)/Q(I+1,3)
        QM(5,I)=XM
        DO 58 J=3,6
    58  Q(I+1,J)=Q(I+1,J)*XM-Q(I,J)
        RETURN
        END
```

```
      SUBROUTINE FIRINT(H)
C     OIL EQUATION
      REAL K,KRW1,KRWN,KRON,KRO1
      COMMON /GROUP1/K(7)
      COMMON /GROUP2/Y1,Y2,Y3,Y4,Y5,Y6
      COMMON /GROUP3/AOLD(50),BOLD(50),ANEW(50),BNEW(50),ASAVER(50),
     1BSAVER(50)
      COMMON /GROUP4/C(50,6),D(50,6),E(50,6),F(50,6),Q(50,6),QM(5,50)
      COMMON /GROUP5/ZO(3),ZW(3),KRW1,KRWN,KRO1,KRON
      COMMON /GROUP6/EPSI,SWI,POR,SOR,AREA,OILVIS,WATVIS,QIN,ABPER,DT,
     1N,ITMAX
      NM3=N-3
      PO=ABPER/OILVIS
      PW=ABPER/WATVIS
      KRO1=ZO(1)+ZO(2)*BOLD(1)+ZO(3)*BOLD(1)**2
      KRW1=ZW(1)+ZW(2)*BOLD(1)+ZW(3)*(BOLD(1)**2)
      KRWN=ZW(1)+ZW(2)*BOLD(N-1)+ZW(3)*(BOLD(N-1)**2)
      KRON=ZO(1)+ZO(2)*BOLD(N-1)+ZO(3)*(BOLD(N-1)**2)
      DO 30 I=3,NM3,2
      I1=I+1
      CALL PERM(BOLD(I-2),BOLD(I-1),BOLD(I),BOLD(I+1),ZO,H)
      C(I,1)=PO*Y1
      C(I,2)=PO*Y2
      C(I,3)=-PO*Y1
      C(I,4)=PO*Y3
      C(I1,1)=-PO*Y3
      C(I1,2)=-PO*Y5
      C(I1,3)=PO*Y3
      C(I1,4)=-PO*Y6
      CALL PERM(BOLD(I),BOLD(I+1),BOLD(I+2),BOLD(I+3),ZO,H)
      C(I,3)=C(I,3)-PO*Y1
      C(I,4)=C(I,4)-PO*Y2
      C(I,5)=PO*Y1
      C(I,6)=-PO*Y3
      C(I1,3)=C(I1,3)-PO*Y2
```

```
      C(I1,4)=C(I1,4)-PO*Y4
      C(I1,5)=PO*Y2
      C(I1,6)=-PO*Y5
   30 CONTINUE
      CALL PERM(BOLD(1),BOLD(2),BOLD(3),BOLD(4),ZO,H)
      C(1,3)=-PO*Y1
      C(1,4)=-PO*Y2-KRO1*PO
      C(1,5)=PO*Y1
      C(1,6)=-PO*Y3
      C(2,3)=-PO*Y2
      C(2,4)=-PO*Y4
      C(2,5)=PO*Y2
      C(2,6)=-PO*Y5
      CALL PERM(BOLD(N-3),BOLD(N-2),BOLD(N-1),BOLD(N),ZO,H)
      C(N-1,1)=PO*Y1
      C(N-1,2)=PO*Y2
      C(N-1,3)=-PO*Y1
      C(N-1,4)=PO*Y3+ABPER*KRON/OILVIS
      C(N,1)=-PO*Y3
      C(N,2)=-PO*Y5
      C(N,3)=PO*Y3
      C(N,4)=-PO*Y6
C
C     WATER EQUATION
      DO 33 I=3,NM3,2
      I1=I+1
      CALL PERM(BOLD(I-2),BOLD(I-1),BOLD(I),BOLD(I+1),ZW,H)
      E(I,1)=PW*Y1
      E(I,2)=PW*Y2
      E(I,3)=-PW*Y1
      E(I,4)=PW*Y3
      E(I1,1)=-PW*Y3
      E(I1,2)=-PW*Y5
      E(I1,3)=PW*Y3
      E(I1,4)=-PW*Y6
```

```
      CALL PERM(BOLD(I),BOLD(I+1),BOLD(I+2),BOLD(I+3),ZW,H)
      E(I,3)=E(I,3)-PW*Y1
      E(I,4)=E(I,4)-PW*Y2
      E(I,5)=PW*Y1
      E(I,6)=-PW*Y3
      E(I1,3)=E(I1,3)-PW*Y2
      E(I1,4)=E(I1,4)-PW*Y4
      E(I1,5)=PW*Y2
      E(I1,6)=-PW*Y5
   33 CONTINUE
      CALL PERM(BOLD(1),BOLD(2),BOLD(3),BOLD(4),ZW,H)
      E(1,3)=-PW*Y1
      E(1,4)=-PW*Y2-ABPER*KRW1/WATVIS
      E(1,5)=PW*Y1
      E(1,6)=-PW*Y3
      E(2,3)=-PW*Y2
      E(2,4)=-PW*Y4
      E(2,5)=PW*Y2
      E(2,6)=-PW*Y5
      CALL PERM(BOLD(N-3),BOLD(N-2),BOLD(N-1),BOLD(N),ZW,H)
      E(N-1,1)=PW*Y1
      E(N-1,2)=PW*Y2
      E(N-1,3)=-PW*Y1
      E(N-1,4)=PW*Y3+ABPER*KRWN/WATVIS
      E(N,1)=-PW*Y3
      E(N,2)=-PW*Y5
      E(N,3)=-PW*Y3
      E(N,4)=-PW*Y6
C
      RETURN
      END
```

```
      SUBROUTINE PERM(B1,B2,B3,B4,Z,H)
      DIMENSION Z(3)
      REAL K
      COMMON /GROUP1/K(7)
C
      K(7)=Z(1)+Z(2)*B1+Z(3)*(B1**2)
      K(1)=Z(2)*H*B2+2.0*Z(3)*B1*B2*H
      K(2)=Z(2)*(-H*B4+3.0*B3-2.*H*B2-3.*B1)+Z(3)*(-6.*(B1**2)
     1*(H*B2)**2-4.*B1*B2*H+6.*B1*B3-2.*B1*B4*H)
      K(3)=Z(2)*(2.*B1+H*B2-2.*B3+H*B4)+Z(3)*(4.*(B1**2)
     1-4.*(H*B2)**2-4.*B1*B2*H-4.*B1*B3+2.*B1*B4*H+6.*B2*B3*H
     2-2.*B2*B4*(H**2))
      K(4)=Z(3)*(9.*(B1**2)+6.*(H*B2)**2+9.*B3**2+(H*B4)**2
     1+16.*B1*B2*H-18.*B1*B3+6.*B1*B4*H-16.*B2*B3*H
     2+6.*B2*B4*(H**2)-6.*B3*B4*H)
      K(5)=Z(3)*(-12.*B1**2-4.*(H*B2)**2-12.*(B3**2)-2.*(H*B4)
     1**2-14.*B1*B2*H+24.*B1*B3-10.*B1*B4*H+14.*B2*B3*H
     2-6.*B2*B4*(H**2)*10.*B3*B4*H)
      K(6)=Z(3)*(4.*B1**2+(H*B2)**2+4.*B3**2+(H*B4)**2
     1+4.*B1*B2*H-8.*B1*B3+4.*B1*B4*H-4.*B2*B3*H+2.*B2*B4
     2*(H**2)-4.*B3*B4*H)
      CALL POLY(H)
      RETURN
      END
```

```
      SUBROUTINE POLY(H)
      REAL K
      COMMON /GROUP1/K(7)
      COMMON /GROUP2/Y1,Y2,Y3,Y4,Y5,Y6
C
      A2=K(7)
      A3=-2.*K(7)+K(1)
      A4=K(7)-2.*K(1)+K(2)
      A5=K(1)-2.*K(2)*K(3)
      A6=K(2)-2.*K(3)+K(4)
      A7=K(3)-2.*K(4)+K(5)
      A8=K(4)-2.*K(5)+K(6)
      A9=K(5)-2.*K(6)
      A10=K(6)
      Y1=36.0*(A2/3.0+A3/4.0+A4/5.0+A5/6.0+A6/7.0+A7/8.0
     1+A8/9.0+A9/10.0+A10/11.0)/H
C
      A1=-K(7)
      A2=5.*K(7)-K(1)
      A3=-7.*K(7)+5.*K(1)=K(2)
      A4=3.*K(7)-7.*K(1)+5.*K(2)-K(3)
      A5=3.*K(1)-7.*K(2)+5.*K(3)-K(4)
      A6=3.*K(2)-7.*K(3)+5.*K(4)-K(5)
      A7=3.*K(3)-7.*K(4)+5.*K(5)-K(6)
      A8=3.*K(4)-7.*K(5)+5.*K(6)
      A9=3.*K(5)-7.*K(6)
      A10=3.*K(6)
      Y2=6.0*(A1/2.0+A2/3.0+A3/4.0+A4/5.0+A5/6.0+A6/7.0+A7/8.0
     1+A8/9.0+A9/10.0+A10/11.0)
C
      A2=2.*K(7)
      A3=-5.*K(7)+2.*K(1)
      A4=3.*K(7)-5.*K(1)+2.*K(2)
      A5=3.*K(1)-5.*K(2)+2.*K(3)
      A6=3.*K(2)-5.*K(3)+2.*K(4)
```

```
      A7=3.*K(3)-5.*K(4)+2.*K(5)
      A8=3.*K(4)-5.*K(5)+2.*K(6)
      A9=3.*K(5)-5.*K(6)
      A10=3.*K(6)
      Y3=6.*(A2/3.+A3./4.+A4/5.+A5/6.+A6/7.+A7/8.+A8/9.
     1+A9/10.+A10/11.)
C
      A0=K(7)
      A1=-8.*K(7)+K(1)
      A2=22.*K(7)-8.*K(1)+K(2)
      A3=-24.*K(7)+22.*K(1)-8.*K(2)+K(3)
      A4=9.*K(7)-24.*K(1)+22.*K(2)-8.*K(3)+K(4)
      A5=9.*K(1)-24.*K(2)+22.*K(3)-8.*K(4)+K(5)
      A6=9.*K(2)-24.*K(3)+22.*K(4)-8.*K(5)+K(6)
      A7=9.*K(3)-24.*K(4)+22.*K(5)-8.*K(6)
      A8=9.*K(4)-24.*K(5)+22.*K(6)
      A9=9.*K(5)-24.*K(6)
      A10=9.*K(6)
      Y4=H*(A0+A1/2.+A2/3.+A3/4.+A4/5.+A5/6.+A6/7.+A7/8.+A8/9.
     1+A9/10.+A10/11.)
C
      A1=-2.*K(7)
      A2=11.*K(7)-2.*K(1)
      A3=-18.*K(7)+11.*K(1)-2.*K(2)
      A4=9.*K(7)-18.*K(1)+11.*K(2)-2.*K(3)
      A5=9.*K(1)-18.*K(2)+11.*K(3)-2.*K(4)
      A6=9.*K(2)-18.*K(3)+11.*K(4)-2.*K(5)
      A7=9.*K(3)-18.*K(4)+11.*K(5)-2.*K(6)
      A8=9.*K(4)-18.*K(5)+11.*K(6)
      A9=9.*K(5)-18.*K(6)
      A10=9.*K(6)
      Y5=H*(A1/2.+A2/3.+A3/4.+A4/5.+A5/6.+A6/7.+A7/8.+A8/9.
     1+A9/10.+A10/11.)
C
      A2=4.*K(7)
```

```
      A3=-12.*K(7)+4.*K(1)
      A4=9.*K(7)-12.*K(1)+4.*K(2)
      A5=9.*K(1)-12.*K(2)+4.*K(3)
      A6=9.*K(2)-12.*K(3)+4.*K(4)
      A7=9.*K(3)-12.*K(4)+4.*K(5)
      A8=9.*K(4)-12.*K(5)+4.*K(6)
      A9=9.*K(5)-12.*K(6)
      A10=9.*K(6)
      Y6=H*(A2/3.+A3/4.+A4/5.+A5/6.+A6/7.+A7/8.+A8/9.+A9/10.
     1+A10/11.)
C
      RETURN
      END
```